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RAYLEIGH-BLOCH WAVE EXPANSIONS FOR DIFFRACTION GRATINGS II.(U)
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Abstract

This report is the second part of a study of Rayleigh-Bloch (R-B) wave expansions for plane diffraction gratings. The principal concepts and results of the study, including the definition and construction of the R-B waves, the formulation of the corresponding R-B wave expansion theorem and the main ideas for the proofs were given in part I. The present part II contains complete proofs of the results whose proofs were omitted or only sketched in part I.

Introduction.

This report is a companion to "Rayleigh-Bloch Wave Expansions for Diffraction Gratings I." In that report the concept of a Rayleigh-Bloch, or R-B, wave for a diffraction grating was introduced and the main concepts and results of a theory of R-B wave expansions were developed. Proofs that were omitted or only sketched in part I are presented in full here. From a technical view-point the central concepts of the theory of R-B wave expansions, as developed in part I, are the holomorphic family $\{A_{p,\zeta,r} : \zeta \in M_p\}$ and the corresponding analytic continuation of the resolvent of the reduced grating propagator A_p. The proofs of the basic properties of $A_{p,\zeta,r}$, described in §4 of part I, are surprisingly intricate and make up the main part of this part II. They are developed in §8. The theorems concerning the R-B wave expansions for the reduced propagator A_p, formulated in §5 of part I, are proved in §9. Finally, the R-B wave expansions for the full grating propagator A, presented in §6 of part I, are proved in §10. References cited with square brackets in this report refer to the list of references at the end of part I.

§8. Proofs of the Results of §4.

The properties of the class $E_{p,z,r}$ described by Lemma 4.1 are essential for the construction of the Riemann surface M_p and family $\{A_{p,\zeta,r} : \zeta \in M_p\}$. The proof of the lemma outlined in §4 is therefore completed here.

Proof of Lemma 4.1. Assume that $u \in D(A_p^{\ell oc})$ and define $v(x,y) = \exp\{-ipx\}\, u(x,y)$. Then $v \in L_2^{1,\ell oc}(\Delta,\Omega)$ and satisfies the p-periodic boundary conditions (3.7) with $p = 0$. Thus if Ω^γ is the cylinder obtained by identifying the points $(-\pi,y)$ and (π,y), $y \in \gamma$, it follows that v is a distribution solution of $\Delta v + 2ip\, D_x v + (z-p^2)v = e^{-ipx} f \in L_2^{\ell oc}(\Omega^\gamma)$. Let h' satisfy $0 < h' < h$, $R_{h'}^2 \subset G$. Such numbers h' exist if $\overline{R_h^2} \subset G$. Then $\Omega_{h'}^\gamma = \Omega^\gamma \cap \{(x,y) \mid y > h'\}$ is contained in the *interior of* Ω^γ and the interior elliptic estimates of [1] imply that $v \in L_2^{2,\ell oc}(\Omega_{h'}^\gamma)$. This result implies (4.6) and (4.8) of Lemma 4.1. Moreover, $f = 0$ in Ω_r and the regularity theory of [1] implies $v \in L_2^{m,\ell oc}(\Omega_r^\gamma)$ for all $m \in Z$ which implies (4.9) and (4.10).

It remains to prove (4.7). Note that if v is defined as above then

$$(8.1)\qquad u_m(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{e^{i(p+m)x}}\, u(x,y)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{e^{imx}}\, v(x,y)dx = v_m(y).$$

Hence (4.7) is equivalent to the statement that

$$(8.2)\qquad v(x,y) = \sum_{m\in Z} v_m(y)\, e^{imx} \text{ in } \Omega_h^\gamma$$

where the series converges to v in $L_2^{2,\ell oc}(\Omega_h^\gamma)$. To prove this note that $\{e^{imx} \mid m \in Z\}$ is an orthogonal sequence in $L_2^2(\Omega_{k,k'}^\gamma)$ for any k, k' such that $h \leq k < k' < \infty$. Next define

$$(8.3) \qquad P_\ell\, v(x,y) = \sum_{|m|\leq\ell} v_m(y)\, e^{imx}$$

where v_m is defined by (8.1). Then direct calculation shows that

$$(8.4) \qquad P_\ell : L_2^2(\Omega_{k,k'}^\gamma) \to L_2^2(\Omega_{k,k'}^\gamma) \text{ is bounded}$$

and

$$(8.5) \qquad P_\ell^2 = P_\ell = P_\ell^* \text{ in } L_2^2(\Omega_{k,k'}^\gamma);$$

i.e., P_ℓ is an orthogonal projection. It follows that

$$(8.6) \qquad Q_\ell = 1 - P_\ell$$

is also an orthogonal projection in $L_2^2(\Omega_{k,k'}^\gamma)$. Note that the convergence of (8.2) in $L_2^{2,\ell oc}(\Omega_{k,k'}^\gamma)$ is equivalent to the condition

$$(8.7) \qquad \lim_{\ell\to\infty} \| Q_\ell\, v\|_2 = 0 \text{ for all } v \in L_2^2(\Omega_{k,k'}^\gamma)$$

where $\|\cdot\|_2$ is the norm in $L_2^2(\Omega_{k,k'}^\gamma)$. Now (8.7) follows from classical convergence theory for Fourier series if $v \in C^\infty(\overline{\Omega_{k,k'}^\gamma})$, the set of restrictions to $\overline{\Omega_{k,k'}^\gamma}$ of functions from $C^\infty(\Omega_h^\gamma)$. Moreover, this set is dense in $L_2^2(\Omega_{k,k'}^\gamma)$. Thus if $v \in L_2^2(\Omega_{k,k'}^\gamma)$ and $v' \in C^\infty(\overline{\Omega_{k,k'}^\gamma})$ then

$$(8.8) \qquad \begin{aligned} \|Q_\ell\, v\|_2^2 &= (Q_\ell\, v,v)_2 = (Q_\ell(v-v'),v)_2 + (Q_\ell\, v',v)_2 \\ &\leq \|v-v'\|_2\, \|v\|_2 + \|Q_\ell\, v'\|_2\, \|v\|_2 \end{aligned}$$

It follows that

$$\text{(8.9)} \qquad \limsup_{\ell\to\infty} \|Q_\ell v\|_2^2 \le \|v-v'\|_2 \|v\|_2$$

for all $v' \in C^\infty(\overline{\Omega^\gamma_{k,k'}})$ which implies (8.7).

Proof of Theorem 4.2. To prove the continuity of the mappings $(p,\zeta) \to w_{p+m}(\zeta)$ for all $(p,\zeta) \in M$ and $m \in Z$ let $(p_0,\zeta_0) \in M$, $m \in Z$ and $\varepsilon > 0$. It will be shown that there exist $\rho_0(\varepsilon) > 0$ and $\delta_0(\varepsilon) > 0$ such that

$$\text{(8.10)} \qquad |w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)| < \varepsilon \text{ for } (p,\zeta) \in N(p_0,\zeta_0,\rho,\delta)$$

provided $0 < \rho \le \rho_0(\varepsilon)$, $0 < \delta \le \delta_0(\varepsilon)$.

To prove (8.10) note that in Cases 1 and 2 of the definition of $N(p_0,\zeta_0,\rho,\delta)$ one has, for every $m \in Z$,

$$\text{(8.11)} \quad w_{p+m}(\zeta) = \pm(\pi_p(\zeta) - (p+m)^2)^{1/2}\,,\; w_{p_0+m}(\zeta_0) = \pm(\pi_{p_0}(\zeta_0) - (p_0+m)^2)^{1/2}$$

where the square roots have non-negative imaginary part and the $\pm$ signs are the same for each $m \in Z$. Moreover,

$$\text{(8.12)} \qquad \pi_p(\zeta),\ \pi_{p_0}(\zeta_0) = z_0 \in D(z_0,\rho) \text{ and } |p - p_0| < \delta$$

for $(p,\zeta) \in N(p_0,\zeta_0,\rho,\delta)$. Hence there exist $\rho_0(\varepsilon) > 0$, $\delta_0(\varepsilon) > 0$ such that

$$\text{(8.13)} \qquad \begin{aligned} &|w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)| \\ &= |(\pi_p(\zeta) - (p+m)^2)^{1/2} - (\pi_{p_0}(\zeta_0) - (p_0+m)^2)^{1/2}| < \varepsilon \end{aligned}$$

for $(p,\zeta) \in N(p_0,\zeta_0,\rho_0(\varepsilon),\delta_0(\varepsilon))$. To prove (8.10) in Case 3 note that in this case if $(p,\zeta) \in N(p_0,\zeta_0,\rho,\delta)$ then one has both $z_0 = (p_0 + m_0)^2$

and $(p + m_0)^2$ in $D(z_0,\rho)$ for $\delta \leq \delta_0(\rho)$. Moreover $w_{p_0+m_0}(\zeta_0) = 0$. Hence there exists a $\rho_0(\varepsilon) > 0$ such that

$$(8.14) \qquad |w_{p+m_0}(\zeta) - w_{p_0+m_0}(\zeta_0)| = |(\pi_p(\zeta) - (p + m_0)^2)^{1/2}| < \varepsilon$$

for $(p,\zeta) \in N(p_0,\zeta_0,\rho,\delta)$, $0 < \rho \leq \rho_0(\varepsilon)$, $0 < \delta \leq \delta_0(\rho_0(\varepsilon))$ because $\pi_p(\zeta) \in D(z_0,\rho)$ for all such (p,ζ). The proof that the functions $w_{p+m}(\zeta)$ with $m \neq m_0$ are continuous at (p_0,ζ_0) is the same as in Case 1.

To prove the equicontinuity statement of Theorem 4.2 fix $(p_0,\zeta_0) \in M$. Then for all $m \in Z$ (resp., $m \in Z - \{m_0\}$) Cases 1 or 2 apply to $w_{p+m}(\zeta)$ and if

$$(8.15) \qquad F_m(z,p) = (z - (p + m)^2)^{1/2} \text{ , Im } F_m(z,p) \geq 0$$

then for all $(p,\zeta) \in N(p_0,\zeta_0,\rho,\delta)$ one has

$$(8.16) \qquad |w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)| = |F_m(\pi_p(\zeta),p) - F_m(\pi_{p_0}(\zeta_0),p_0)|.$$

Note that $F_m(z,p)$ has partial derivatives

$$(8.17) \qquad \begin{aligned} D_z F_m(z,p) &= \frac{1}{2} (z - (p + m)^2)^{-1/2} , \\ D_p F_m(z,p) &= -(z - (p + m)^2)^{-1/2} (p + m). \end{aligned}$$

Hence for $z \in D(z_0,\rho)$ and $|p - p_0| < \delta$ these derivatives are uniformly bounded for all $m \in Z$ (resp., $m \in Z - \{m_0\}$). Now by Taylor's theorem

$$(8.18) \quad F_m(z,p) = F_m(z_0,p_0) + (z - z_0) D_z F_m(z',p') + (p - p_0) D_p F_m(z',p')$$

where (z',p') is on the segment from (z_0,p_0) to (z,p). Thus one has

$$|w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)|$$

(8.19)

$$\leq |D_z F_m(z',p')| \; |\pi_p(\zeta) - \pi_{p_0}(\zeta_0)| + |D_p F_m(z',p')| \; |p - p_0|$$

$$\leq \text{Const.} \; (|\pi_p(\zeta) - \pi_{p_0}(\zeta_0)| + |p - p_0|)$$

for all $(p,\zeta) \in N(p_0,\zeta_0,\rho,\delta)$ and all $m \in Z$ (resp., $m \in Z - \{m_0\}$). Since $\pi_p(\zeta)$ and $\pi_{p_0}(\zeta_0) = z_0$ are in $D(z_0,\rho)$ for $(p,\zeta) \in N(p_0,\zeta_0,\rho,\delta)$ it is clear that there exist $\rho_0(\varepsilon) > 0$, $\delta_0(\varepsilon) > 0$ such that (8.14) holds for all (p,ζ) in $N(p_0,\zeta_0,\rho_0(\varepsilon),\delta_0(\varepsilon))$ and all $m \in Z$.

Theorem 4.3 was proved in §4. The proof of Theorem 4.4 will be based on the following

Lemma 8.1. *For every compact set* $K \subset M$ *and every* $r' > r$ *there exists a constant* $C_1 = C_1(K,r,r')$ *such that for all* $u \in \bigcup_{(p,\zeta)\in K} F_{p,\zeta,r}$ *one has*

$$\|u\|^2_{r,r'} \leq C_1(\|u\|^2_{h,r} + \|\nabla u\|^2_{h,r}). \tag{8.20}$$

Proof of Lemma 8.1. Note that every $u \in F_{p,\zeta,r}$ can be written

$$u(X) = u'(X) + u''(X), \; X \in \Omega_r, \tag{8.21}$$

where

$$u'(X) = \Sigma' \; c_m \exp\{i\,x(p + m) + i\,y\,w_{p+m}(\zeta)\}, \tag{8.22}$$

$$u''(X) = \Sigma'' \; c_m \exp\{i\,x(p + m) + i\,y\,w_{p+m}(\zeta)\}, \tag{8.23}$$

the c_m are the coefficients of (4.28) and the notation Σ', Σ'' denotes summation over the index sets $\{m \mid \operatorname{Im} w_{p+m}(\zeta) > 0\}$ and $\{m \mid \operatorname{Im} w_{p+m}(\zeta) \leq 0\}$, respectively. Lemma 4.1 implies that $u \in C^\infty(\overline{\Omega}_r)$ and the Fourier series in (8.22) and its derivatives converge uniformly on compact subsets of $\overline{\Omega}_r$ to u' and its derivatives. Moreover, the sum in (8.23) is finite for each $(p,\zeta) \in M$. Finally

$$\|u\|^2_{r,r'} = \|u'\|^2_{r,r'} + \|u''\|^2_{r,r'} \tag{8.24}$$

because $\{e^{i(p+m)x}\}$ is an orthogonal sequence in $L_2(\Omega_{r,r'})$ and the index sets defining Σ' and Σ'' are complementary.

Parseval's relation for Fourier series implies that

$$\int_{-\pi}^{\pi} |u'(x,y)|^2 \, dx = 2\pi \, \Sigma' \, |c_m|^2 \exp\{-2y \operatorname{Im} w_{p+m}(\zeta)\} \tag{8.25}$$

for all $y \geq r$. Moreover, this is a monotone decreasing function of y, whence

$$\begin{aligned} \int_{-\pi}^{\pi} |u'(x,y)|^2 \, dx &\leq 2\pi \, \Sigma' \, |c_m|^2 \exp\{-2r \operatorname{Im} w_{p+m}(\zeta)\} \\ &= 2\pi \, \Sigma' \, |u_m(r)|^2 . \end{aligned} \tag{8.26}$$

Integrating this inequality over $r \leq y \leq r'$ gives

$$\|u'\|^2_{r,r'} \leq 2\pi(r' - r) \, \Sigma' \, |u_m(r)|^2 . \tag{8.27}$$

The analogue of (8.25) for u'' is a monotone increasing function of $y \geq r$. In particular, for $r \leq y \leq r'$ one has

$$\int_{-\pi}^{\pi} |u''(x,y)|^2 \, dx \leq 2\pi \, \Sigma'' \, |c_m|^2 \exp\{-2r' \operatorname{Im} w_{p+m}(\zeta)\} . \tag{8.28}$$

To estimate this sum note that the sets $\{m \mid \operatorname{Im} w_{p+m}(\zeta) \leq 0\}$ vary with $(p,\zeta) \in M$ and the properties of M established in §4 imply that the set

$$\text{(8.29)} \qquad M = M(K) = \bigcup_{(p,\zeta)\in K} \{m \mid \operatorname{Im} w_{p+m}(\zeta) \leq 0\}$$

is finite for each compact $K \subset M$. It follows from this and Theorem 4.2 that

$$\text{(8.30)} \qquad \mu = \mu(K) = \operatorname{Max} \{-\operatorname{Im} w_{p+m}(\zeta) : (p,\zeta) \in K \text{ and } m \in M(K)\}$$

is finite. Hence (8.28) implies

$$\text{(8.31)} \qquad \begin{aligned} \int_{-\pi}^{\pi} |u''(x,y)|^2\, dx &\leq 2\pi \exp\{2(r'-r)\mu\} \,\Sigma''\, |c_m|^2 \exp\{-2r \operatorname{Im} w_{p+m}(\zeta)\} \\ &= 2\pi \exp\{2(r'-r)\mu\} \,\Sigma''\, |u_m(r)|^2 \end{aligned}$$

for $r \leq y \leq r'$. Integrating (8.31) over $r \leq y \leq r'$ gives

$$\text{(8.32)} \qquad \|u''\|^2_{r,r'} \leq 2\pi\, (r'-r) \exp\{2(r'-r)\mu\} \,\Sigma''\, |u_m(r)|^2$$

Adding (8.27) and (8.32) and using (8.24) gives

$$\text{(8.33)} \qquad \|u\|^2_{r,r'} \leq 2\pi(r'-r) \exp\{2(r'-r)\mu\} \sum_{m\in Z} |u_m(r)|^2$$

Finally, Parseval's relation in $L_2(-\pi,\pi)$ gives

$$\text{(8.34)} \qquad \|u(\cdot,r)\|^2 = \int_{-\pi}^{\pi} |u(x,r)|^2\, dx = 2\pi \sum_{m\in Z} |u_m(r)|^2,$$

whence

$$\|u\|^2_{r,r'} \le (r'-r)\exp\{2(r'-r)\mu\}\,\|u(\cdot,r)\|^2. \tag{8.35}$$

To complete the proof of (8.20) recall that by Lemma 4.1, $u \in L_2^{2,\ell oc}(\Omega_h)$. It follows by Sobolev's imbedding theorem [1, p. 32] that there exists a constant $C_2 = C_2(h,r)$ such that

$$\begin{aligned} \|u(\cdot,r)\|^2 &\le C_2(\|u\|^2_{h,r} + \|D_y\, u\|^2_{h,r}) \\ &\le C_2(\|u\|^2_{h,r} + \|\nabla u\|^2_{h,r}) \end{aligned} \tag{8.36}$$

Combining (8.35) and (8.36) gives (8.20).

<u>Proof of Theorem 4.4</u>. It must be shown that there exists a constant $C = C(K,r,r')$ such that for all $(p,\zeta) \in K$ and all $u \in F_{p,\zeta,r}$

$$\|u\|^2_{o,r'} + \|\nabla u\|^2_{o,r'} + \|\Delta u\|^2_{o,r'} \le C^2(\|u\|^2_{o,r} + \|\nabla u\|^2_{o,r} + \|\Delta u\|^2_{o,r}). \tag{8.37}$$

Clearly it will suffice to show that

$$\|u\|^2_{r,r'} + \|\nabla u\|^2_{r,r'} + \|\Delta u\|^2_{r,r'} \le C^2(\|u\|^2_{o,r} + \|\nabla u\|^2_{o,r} + \|\Delta u\|^2_{o,r}) \tag{8.38}$$

since (8.37) then follows with $C^2 + 1$ instead of C^2. Moreover, every $u \in F_{p,\zeta,r}$ satisfies $\Delta u = -\pi(\zeta)u$ in Ω_r. Hence it will suffice to show that

$$\|u\|^2_{r,r'} + \|\nabla u\|^2_{r,r'} \le C^2(\|u\|^2_{o,r} + \|\nabla u\|^2_{o,r} + \|\Delta u\|^2_{o,r}) \tag{8.39}$$

since (8.38) then follows with $C^2 \operatorname{Max}\{|\pi_p(\zeta)| + 1 : (p,\zeta) \in K\}$ instead of C^2.

To prove (8.39) note that the Fourier series argument used in the proof of Lemma 8.1 implies that (cf. (8.35))

$$(8.40) \qquad \|\nabla u\|^2_{r,r'} \leq (r'-r) \exp\{2(r'-r)\mu\} \|\nabla u(\cdot,r)\|^2 .$$

Moreover, if $r'' = \frac{1}{2}(h + r)$ then $h < r'' < r$ and Sobolev's imbedding theorem implies that there exists a constant $C_3 = C_3(h,r)$ such that

$$(8.41) \qquad \|\nabla u(\cdot,r)\|^2 \leq C_3 \|u\|^2_{2;r'',r}$$

where $\|\cdot\|_{2;r'',r}$ is the norm for $L^2_2(\Omega_{r'',r})$. Finally, the interior elliptic estimates of [1], applied to $v(x,y) = \exp\{-ipx\} u(x,y)$ and $L_p v = \Delta v + 2ip D_x v - p^2 v$ in $\Omega^\gamma_{h,r'}$, imply that there exists a constant $C_4 = C_4(h,r,r')$ such that

$$(8.42) \qquad \|u\|^2_{2;r'',r} \leq C_4 (\|u\|^2_{h,r'} + \|\Delta u\|^2_{h,r'}).$$

Moreover, since $\Delta u = -\pi(\zeta)u$ in $\Omega_{r,r'}$

$$(8.43) \qquad \begin{aligned} \|u\|^2_{h,r'} + \|\Delta u\|^2_{h,r'} &= \|u\|^2_{h,r} + \|u\|^2_{r,r'} + \|\Delta u\|^2_{h,r} + |\pi_p(\zeta)|^2 \|u\|^2_{r,r'} \\ &\leq \|u\|^2_{h,r} + \|\Delta u\|^2_{h,r} + C_5(K) \|u\|^2_{r,r'} \end{aligned}$$

where $C_5(K) = \text{Max}\{|\pi_p(\zeta)|^2 : (p,\zeta) \in K\}$. Combining (8.20), (8.40), (8.41), (8.42) and (8.43) gives

$$(8.44) \qquad \|u\|^2_{r,r'} + \|\nabla u\|^2_{r,r'} \leq C_6(\|u\|^2_{h,r} + \|\nabla u\|^2_{h,r} + \|\Delta u\|^2_{h,r}) + C_7 \|u\|^2_{r,r'}$$

where $C_6 = \text{Max}(C_1, (r'-r)\exp\{2(r'-r)\mu(K)\} C_3 C_4)$ and $C_7 = (r'-r)\exp\{2(r'-r)\mu(K)\} C_3 C_4 C_5$. Finally, combining (8.20) and (8.44) gives (8.39) with $C^2 = \text{Max}(C_6, C_1 C_7)$.

It is worth remarking that an indirect (non-constructive) proof of Theorem 4.4 can be given by a compactness argument; see [34, Lemma 4.6] and Alber [3, Lemma 5.3].

The Sesquilinear Form $A_{p,\zeta,r}$ in $L_2(\Omega_{0,r})$. Kato's first representation theorem [17, p. 322] associates a unique m-sectorial operator in $L_2(\Omega_{0,r})$ with each densely defined, closed, sectorial sesquilinear form in $L_2(\Omega_{0,r})$. Theorem 4.5 will be proved by constructing such a form $A_{p,\zeta,r}$ in $L_2(\Omega_{0,r})$ and showing that $A_{p,\zeta,r}$ is the associated m-sectorial operator. To motivate the definition of $A_{p,\zeta,r}$ note that if $v \in D(A_{p,\zeta,r})$ then application of Green's theorem gives

$$(8.45) \qquad (v, A_{p,\zeta,r} v)_{0,r} = \|\nabla v\|^2_{0,r} - \int_{-\pi}^{\pi} \bar{v}\, D_y v \Big|_{y=r} dx.$$

The formal correctness of this equation is obvious. A rigorous proof based on the definition of $A_p^{\ell oc}$ is given below; see (8.115). Now $v = P_{p,\zeta,r}\, u$ where $u \in F_{p,\zeta,r}$ and u and v have Fourier expansions (4.7) for $h \le y < \infty$ and $h \le y \le r$, respectively. Moreover, Lemma 4.1 and the Sobolev theorems [1] imply that $u_m \in C^1[h,\infty)$, $v_m \in C^1[h,r]$, $u_m(y) = v_m(y)$ for $h \le y \le r$ and

$$(8.46) \qquad u_m(y) = c_m \exp\{i\, y\, w_{p+m}(\zeta)\}, \quad y \ge r.$$

Application of Parseval's formula to the integral in (8.45) gives the alternative representation

$$(8.47) \qquad (v, A_{p,\zeta,r} v)_{0,r} = \|\nabla v\|^2_{0,r} - 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\, |v_m(r)|^2.$$

The right-hand side of (8.47) will be used to define the form $A_{p,\zeta,r}$. Two cases, corresponding to the Dirichlet and Neumann boundary conditions respectively, must be distinguished. To this end define

$$G^{D}_{p,\zeta,r}$$

(8.48)

$$= L_2^{D,p,\ell oc}(\Omega) \cap \{u \mid \text{supp } (\Delta + \pi(\zeta))u \subset \Omega_{0,r};\ (4.28) \text{ holds in } L_2^{1,\ell oc}(\Omega_r)\},$$

$$G^{N}_{p,\zeta,r}$$

(8.49)

$$= L_2^{1,p,\ell oc}(\Omega_{0,r}) \cap \{u \mid \text{supp } (\Delta + \pi(\zeta))u \subset \Omega_{0,r};\ (4.28) \text{ holds in } L_2^{1,\ell oc}(\Omega_r)\}.$$

The condensed notation $G_{p,\zeta,r}$ will be used to denote $G^{D}_{p,\zeta,r}$ or $G^{N}_{p,\zeta,r}$ in statements that hold for both. It is easy to verify that $G_{p,\zeta,r}$ is a Fréchet subspace of $L_2^{1,\ell oc}(\Omega)$. The notation $Q_{p,\zeta,r} : G_{p,\zeta,r} \to L_2(\Omega_{0,r})$ will be used for the natural projection defined by

$$(8.50) \qquad Q_{p,\zeta,r}u = u\Big|_{\Omega_{0,r}} \text{ for all } u \in G_{p,\zeta,r}.$$

The sesquilinear form $A_{p,\zeta,r}$ ($= A^{D}_{p,\zeta,r}$ or $A^{N}_{p,\zeta,r}$) and corresponding quadratic form are defined by

$$(8.51) \qquad D(A_{p,\zeta,r}) = Q_{p,\zeta,r}\, G_{p,\zeta,r} \subset L_2(\Omega_{0,r}),$$

$$(8.52) \qquad A_{p,\zeta,r}(v,v') = (\nabla v, \nabla v')_{0,r} - 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\, \overline{v_m(r)}\, v'_m(r)$$

for all $v,v' \in D(A_{p,\zeta,r})$, and

$$(8.53) \qquad A_{p,\zeta,r}(v) = A_{p,\zeta,r}(v,v),\ v \in D(A_{p,\zeta,r}),$$

and one has

Theorem 8.2. $A_{p,\zeta,r}$ is a densely defined, sectorial, closed sesquilinear form in $L_2(\Omega_{0,r})$.

The proof of this result requires a number of estimates which will be developed in a series of lemmas. The first lemma shows that (8.52) does indeed define a sesquilinear form on $L_2(\Omega_{0,r})$.

Lemma 8.3. For all $v,v' \in D(A_{p,\zeta,r})$ the series in (8.52) converges absolutely.

Proof of Lemma 8.3. It follows from Schwarz's inequality that it will suffice to prove that

$$\sum_{m\in Z} w_{p+m}(\zeta)\ |v_m(r)|^2 \tag{8.54}$$

converges absolutely when $v \in D(A_{p,\zeta,r})$ and

$$v_m(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(p+m)x}\ v(x,r)\ dx. \tag{8.55}$$

To this end write $v = Q_{p,\zeta,r}u$ where $u \in G_{p,\zeta,r}$ and decompose u into

$$u(X) = u'(X) + u''(X),\ X \in \Omega_r, \tag{8.56}$$

as in the proof of Lemma 8.1.

Consider first the component u'. Parseval's relation implies that

$$\int_{-\pi}^{\pi} |u'(x,y)|^2\ dx = 2\pi\ \Sigma'\ |c_m|^2 \exp\{-2y \operatorname{Im} w_{p+m}(\zeta)\}$$

(8.57)

$$\int_{-\pi}^{\pi} |\nabla u'(x,y)|^2\ dx = 2\pi\ \Sigma' |c_m|^2 (|p+m|^2 + |w_{p+m}(\zeta)|^2) \exp\{-2y \operatorname{Im} w_{p+m}(\zeta)\}$$

for all $y \geq r$. Moreover, these are monotone decreasing functions of y that tend to zero exponentially at ∞. Hence $u' \in L_2^1(\Omega_r)$.

Next let $n, n' \in Z$ satisfy $n < n'$ and define

$$(8.58) \qquad u'_{n,n'}(X) = \sum_{n}^{n'}{}' \; c_m \exp \{i\, x(p+m) + i\, y\, w_{p+m}(\zeta)\}$$

where $\sum_{n}^{n'}{}'$ denotes summation over the index set $\{m \mid \operatorname{Im} w_{p+m}(\zeta) > 0$ and $n \leq m \leq n'\}$. Applying Green's theorem to $u'_{n,n'}$ and $\overline{u'_{n,n'}}$ in $\Omega_{r,r'}$ gives

$$(8.59) \qquad \int_{\Omega_{r,r'}} \{\overline{u'_{n,n'}}\, \Delta u'_{n,n'} + |\nabla u'_{n,n'}|^2\}\, dX = \int_{\partial\Omega_{r,r'}} \overline{u'_{n,n'}}\, D_\nu\, u'_{n,n'}\, ds$$

whence, using the Helmholtz equation and p-periodic boundary condition for $u'_{n,n'}$, one has

$$(8.60) \qquad \|\nabla u'_{n,n'}\|^2_{r,r'} - \pi(\zeta)\, \|u'_{n,n'}\|^2_{r,r'} = \int_{-\pi}^{\pi} [\overline{u'_{n,n'}}\, D_y\, u'_{n,n'}]_r^{r'} dx.$$

Making $r' \to \infty$ and writing $\|\cdot\|_r = \|\cdot\|_{r,\infty}$ gives

$$(8.61) \qquad \begin{aligned} \|\nabla u'_{n,n'}\|^2_r - \pi(\zeta)\, \|u'_{n,n'}\|^2_r &= -\int_{-\pi}^{\pi} \overline{u'_{n,n'}}\, D_y\, u'_{n,n'}\Big|_{y=r} dx \\ &= -2\pi i \sum_{n}^{n'}{}' \; w_{p+m}(\zeta)\, |u_m(r)|^2 \end{aligned}$$

where $v_m(r) = u_m(r) = c_m \exp\{i\, r\, w_{p+m}(\zeta)\}$. In particular, taking the real part of (8.61) gives

$$(8.62) \qquad 2\pi \sum_{n}^{n'}{}' \; \operatorname{Im} w_{p+m}(\zeta)\, |v_m(r)|^2 = \|\nabla u'_{n,n'}\|^2_r - \operatorname{Re} \pi(\zeta)\, \|u'_{n,n'}\|^2_r.$$

Hence the convergence of the Fourier series for u' in $L_2^1(\Omega_r)$ implies that

(8.63) $$\Sigma' \text{ Im } w_{p+m}(\zeta) \; |v_m(r)|^2 < \infty.$$

The convergence is absolute because all the terms are non-negative.

Now consider the set

(8.64) $$\{-i \; w_{p+m}(\zeta) \; | \; \text{Im } w_{p+m}(\zeta) > 0\}.$$

Each member of the set satisfies

(8.65) $$|\arg \; (-i \; w_{p+m}(\zeta))| < \pi/2 \; .$$

Moreover, elements of the set (8.64) satisfy

(8.66) $$w_{p+m}(\zeta) \sim i \; |p+m|, \; |m| \to \infty,$$

whence

(8.67) $$\arg \; (-i \; w_{p+m}(\zeta)) \to 0 \text{ when } |m| \to \infty.$$

It follows that

(8.68) $$\theta = \text{Max } \{|\arg \; (-i \; w_{p+m}(\zeta))| \; : \; \text{Im } w_{p+m}(\zeta) > 0\} < \pi/2.$$

Hence if Im $w_{p+m}(\zeta) > 0$ then

(8.69) $$|\text{Re } w_{p+m}(\zeta)| \; |v_m(r)|^2 \leq \tan\theta \text{ Im } w_{p+m}(\zeta) \; |v_m(r)|^2$$

and (8.63) implies that

(8.70) $$\Sigma' \; |\text{Re } w_{p+m}(\zeta)| \; |v_m(r)|^2 < \infty.$$

(8.63), (8.70) and the finiteness of the sum defining u" imply the absolute convergence of the series (8.54).

Lemma 8.4. For each compact $K \subset M$ and each $r > h$ there exists an $a = a(K,r)$ such that for all $v \in \cup_{(p,\zeta)\in K} D(A_{p,\zeta,r})$ one has

$$(8.71) \qquad |2\pi\, \Sigma''\, w_{p+m}(\zeta)\, |v_m(r)|^2| \le \frac{1}{2}\, \|\nabla v\|^2_{0,r} + a\, \|v\|^2_{0,r}.$$

Proof of Lemma 8.4. Schwarz's inequality and the definition (8.55) imply that $2\pi\, |v_m(r)|^2 \le \|v(\cdot,r)\|^2_{L_2(-\pi,\pi)}$. Since $v \in L_2^1(\Omega_{0,r})$ it follows by Sobolev's imbedding theorem [1] that there exists a constant $C_0 = C_0(r)$ such that for all $\varepsilon \ge 1$ one has

$$(8.72) \qquad 2\pi\, |v_m(r)|^2 \le C_0\, \varepsilon^{-1}(\|\nabla v\|^2_{0,r} + \varepsilon^2\, \|v\|^2_{0,r}).$$

Next, note that if $M(K)$ is the index set defined by (8.29) then $M(K)$ is finite and hence

$$(8.73) \qquad C_1 = C_1(K) = \mathrm{Max}\, \{|w_{p+m}(\zeta)| : (p,\zeta) \in K \text{ and } m \in M(K)\}$$

is finite for every compact $K \subset M$. Combining (8.72) and (8.73) gives

$$(8.74) \qquad \begin{aligned} |2\pi\, \Sigma''\, w_{p+m}(\zeta)\, |v_m(r)|^2| &\le 2\pi\, C_1\, \Sigma''\, |v_m(r)|^2 \\ &\le 2\pi\, C_1 \sum_{m\in M} |v_m(r)|^2 \\ &\le C_0\, C_1\, M\, \varepsilon^{-1}(\|\nabla v\|^2_{0,r} + \varepsilon^2 \|v\|^2_{0,r}) \\ &\le \frac{1}{2}\, \|\nabla v\|^2_{0,r} + a\, \|v\|^2_{0,r} \end{aligned}$$

provided that $\varepsilon = \varepsilon(K,r) \ge 1$ is chosen such that $C_0\, C_1\, M\, \varepsilon^{-1} \le 1/2$ and $a = a(K,r)$ satisfies $a \ge C_0\, C_1\, M\, \varepsilon$.

Corollary 8.5. The sesquilinear form $A_{p,\zeta,r}$ is sectorial for all $(p,\zeta) \in M$. In fact, for each compact $K \subset M$ there exist constants

$\gamma = \gamma(K) \in R$ and $\theta = \theta(K) < \pi/2$ such that for all $(p,\zeta) \in K$ and all $v \in D(A_{p,\zeta,r})$ with $\|v\|_{0,r} = 1$ one has

$$A_{p,\zeta,r}(v) \in \{z \in C : |\arg (z - \gamma)| \leq \theta\}. \tag{8.75}$$

Proof of Corollary 8.5. The proof generalizes one of Alber [3, Lemma 6.3]. Let $(p,\zeta) \in K$, $v \in D(A_{p,\zeta,r})$, $\|v\|_{0,r} = 1$ and write $A_{p,\zeta,r}(v) = I_1 + I_2$ where

$$I_1 = \|\nabla v\|^2_{0,r} - 2\pi i\ \Sigma''\ w_{p+m}(\zeta)\ |v_m(r)|^2\ , \tag{8.76}$$

and

$$I_2 = -2\pi i\ \Sigma'\ w_{p+m}(\zeta)\ |v_m(r)|^2. \tag{8.77}$$

Then by Lemma 8.4 one has

$$|\operatorname{Im} I_1| \leq \frac{1}{2} \|\nabla v\|^2_{0,r} + a \tag{8.78}$$

Similarly, the real part of I_1 satisfies

$$\operatorname{Re} I \geq \|\nabla v\|^2_{0,r} - \frac{1}{2} \|\nabla v\|^2_{0,r} - a = \frac{1}{2} \|\nabla v\|^2_{0,r} - a \tag{8.79}$$

Combining (8.78) and (8.79) gives $|\operatorname{Im} I_1| \leq \operatorname{Re} I_1 + 2a$ whence

$$I_1 \in \{z \in C : |\arg (z + 2a)| \leq \pi/4. \tag{8.80}$$

Next, recall that $|\arg I_2| \leq \theta < \pi/2$ where $\theta = \theta(p,\zeta)$ is defined by (8.68). In fact, it is elementary to show that the limit relations (8.66), (8.67) hold uniformly for $(p,\zeta) \in K$ and hence there exists a $\theta_1 = \theta_1(K) < \pi/2$ such that $|\arg I_2| \leq \theta_1$ for all $(p,\zeta) \in K$. Combining this estimate with (8.80) gives (8.75) with $\gamma = -2a$ and $\theta = \operatorname{Max}(\pi/4,\theta_1)$.

The proof that the form $A_{p,\zeta,r}$ is closed is based on the following generalization of an estimate of Alber [3, p. 269].

Theorem 8.6. For each $(p,\zeta) \in M$ and each $r' > r > h$ there exists a constant $C = C(p,\zeta,r,r')$ such that for all $v = Q_{p,\zeta,r}u$ with $u \in G_{p,\zeta,r}$ one has (see (4.30), (4.31) for notation)

$$\|u\|^2_{1;0,r'} \le C(|A_{p,\zeta,r}(v)| + \|v\|^2_{0,r}). \tag{8.81}$$

The proof of Theorem 8.6 will be based on a number of related estimates which will be developed in a series of subsidiary lemmas. The first is

Lemma 8.7. Under the hypotheses of Theorem 8.6 one has

$$\|\nabla u'\|^2_r - \pi(\zeta)\,\|u'\|^2_r = -2\pi i\, \Sigma'\, w_{p+m}(\zeta)\, |u_m(r)|^2 \tag{8.82}$$

where u' by (8.22) and $u_m(y) = c_m \exp\{i\,y\,w_{p+m}(\zeta)\}$.

Proof of Lemma 8.7. The finiteness of the norms in (8.82) has already been noted; see (8.57). Passage to the limit $n \to -\infty$, $n' \to \infty$ in (8.61) gives (8.82).

Lemma 8.8. Under the hypotheses of Theorem 8.6 there exists a constant $C_1 = C_1(p,\zeta,r,r')$ such that for all $u \in G_{p,\zeta,r}$ one has

$$\|u\|^2_{r,r'} \le C_1\, \|u(\cdot,r)\|^2. \tag{8.83}$$

Proof of Lemma 8.8. (8.83) follows from the proof of Lemma 8.1, inequality (8.35).

Lemma 8.9. Under the hypotheses of Theorem 8.6 there exists a constant $C_2 = C_2(p,\zeta,r,r')$ such that for all $u \in G_{p,\zeta,r}$ one has

$$\|\nabla u''\|^2_{r,r'} \le C_2\, \|u(\cdot,r)\|^2. \tag{8.84}$$

The proof of Lemma 8.9, starting from (8.57), is exactly like that of Lemma 8.8 and is therefore omitted.

Lemma 8.10. Under the hypotheses of Theorem 8.6 there exists a constant $C_3 = C_3(p,\zeta,r,r')$ such that for all $u \in G_{p,\zeta,r}$ one has

$$(8.85) \qquad \left|\Sigma'' \, w_{p+m}(\zeta) \, |u_m(r)|^2\right| \leq C_3 \, \|u(\cdot,r)\|^2.$$

Proof of Lemma 8.10. One may take

$C_3 = \mathrm{Max}\,\{|w_{p+m}(\zeta)| : \mathrm{Im}\, w_{p+m}(\zeta) \leq 0\}$ and use (8.34).

Lemma 8.11. Under the hypotheses of Theorem 8.6 there exists a constant $C_4 = C_4(p,\zeta,r,r')$ such that one has

$$(8.86) \qquad \|u'\|_r^2 \leq C_4 \, \|u(\cdot,r)\|^2.$$

Proof of Lemma 8.11. Integration of (8.57) over $r \leq y < \infty$ gives

$$(8.87) \qquad \|u'\|_r^2 = 2\pi \, \Sigma' \, |c_m|^2 \exp\{-2r \, \mathrm{Im}\, w_{p+m}(\zeta)\}/2 \, \mathrm{Im}\, w_{p+m}(\zeta)$$

which with (8.34) implies (8.86) with $C_4(p,\zeta,r,r')$ defined by $C_4^{-1} = \mathrm{Min}\,\{2\, \mathrm{Im}\, w_{p+m}(\zeta) : \mathrm{Im}\, w_{p+m}(\zeta) > 0\}$. This minimum is positive because $\mathrm{Im}\, w_{p+m}(\zeta) \sim |p+m|$, $|m| \to \infty$ (see (8.66)).

Lemma 8.12. Under the hypotheses of Theorem 8.6 to each $\alpha > 0$ there corresponds a constant $\theta_\alpha = \theta_\alpha(h,r)$ such that for all $u \in G_{p,\zeta,r}$ one has

$$(8.88) \qquad \|u(\cdot,r)\|^2 \leq \alpha \, \|\nabla u\|_{0,r}^2 + \theta_\alpha \, \|u\|_{0,r}^2.$$

Proof of Lemma 8.12. Recall that $u \in C^\infty(\overline{\Omega}_r)$ and hence $u_m(y) \in C^\infty[r,\infty)$. Hence, by a Sobolev inequality there is a constant $\gamma = \gamma(h,r)$ such that for all $\varepsilon \geq 1$ one has [1]

$$(8.89) \qquad |u_m(r)|^2 \le \gamma\, \varepsilon^{-1} \left(\int_h^r |u_m(y)|^2\, dy + \varepsilon^2 \int_h^r |u_m(y)|^2\, dy\right)$$

Moreover, by Parseval's relation,

$$(8.90) \qquad \|u(\cdot,y)\|^2 = \int_{-\pi}^{\pi} |u(x,y)|^2\, dx = 2\pi \sum_{m\in Z} |u_m(y)|^2, \; y \ge h,$$

whence

$$(8.91) \qquad \|u\|_{h,r}^2 = 2\pi \sum_{m\in Z} \int_h^r |u_m(y)|^2\, dy,$$

$$(8.92) \qquad \|D_y u\|_{h,r}^2 = 2\pi \sum_{m\in Z} \int_h^r |D_y\, u_m(y)|^2\, dy.$$

Combining (8.89) - (8.92) gives the estimate

$$(8.93) \qquad \begin{aligned} \|u(\cdot,r)\|^2 &\le \gamma\, \varepsilon^{-1}(\|D_y u\|_{h,r}^2 + \varepsilon^2\, \|u\|_{h,r}^2) \\ &\le \gamma\, \varepsilon^{-1}(\|\nabla u\|_{0,r}^2 + \varepsilon^2\, \|u\|_{0,r}^2) \end{aligned}$$

Choosing $\gamma(h,r)\varepsilon^{-1} = \alpha$, $\gamma(h,r)\varepsilon = \gamma^2(h,r)/\alpha = \theta_\alpha(h,r)$ in (8.93) gives (8.88).

Proof of Theorem 8.6. The definition (8.52), (8.53) implies that for all $v = Q_{p,\zeta,r} u \in D(A_{p,\zeta,r})$

$$(8.94) \qquad A_{p,\zeta,r}(v) = \|\nabla v\|_{0,r}^2 - 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\; |u_m(r)|^2.$$

Combining this with Lemma 8.7 gives the representation

$$(8.95) \qquad A_{p,\zeta,r}(v) = \|\nabla v\|^2_{0,r} + \|\nabla u'\|^2_r - \pi(\zeta)\ \|u'\|^2_r - 2\pi i\ \Sigma'' w_{p+m}(\zeta)\,|u_m(r)|^2$$

whence

$$(8.96) \qquad \begin{aligned} \|\nabla v\|^2_{0,r} + \|\nabla u'\|^2_{r,r'} &\leq \|\nabla v\|^2_{0,r} + \|\nabla u'\|^2_r \\ &= \mathrm{Re}\ \{A_{p,\zeta,r}(v) + \pi(\zeta)\|u'\|^2_r + 2\pi i\ \Sigma'' w_{p+m}(\zeta)\ |u_m(r)|^2\} \\ &\leq |A_{p,\zeta,r}(v)| + |\pi(\zeta)|\ \|u'\|^2_r + 2\pi\ |\Sigma'' w_{p+m}(\zeta)\ |u_m(r)|^2| \end{aligned}$$

It follows that

$$(8.97) \qquad \begin{aligned} \|u\|^2_{1;0,r'} &= \|\nabla v\|^2_{0,r} + \|\nabla u'\|^2_{r,r'} + \|\nabla u''\|^2_{r,r'} + \|v\|^2_{0,r} + \|u\|^2_{r,r'} \\ &\leq |A_{p,\zeta,r}(v)| + |\pi(\zeta)|\ \|u'\|^2_r + 2\pi\ |\Sigma''\ w_{p+m}(\zeta)\ |u_m(r)|^2| \\ &\quad + \|\nabla u''\|^2_{r,r'} + \|u\|^2_{r,r'} + \|v\|^2_{0,r} \end{aligned}$$

Combining (8.97) and the estimates of Lemmas 8.8 - 8.11 gives

$$(8.98) \qquad \|u\|^2_{1;0,r'} \leq |A_{p,\zeta,r}(v)| + C_5\ \|u(\cdot,r)\|^2 + \|v\|^2_{0,r}$$

where

$$(8.99) \qquad C_5 = C_5(p,\zeta,r,r') = C_1 + C_2 + 2\pi\ C_3 + |\pi(\zeta)|\ C_4.$$

On combining (8.98) and (8.88), and recalling that $u = v$ in $\Omega_{0,r}$, one finds

$$(8.100) \qquad \|u\|^2_{1;0,r'} \leq |A_{p,\zeta,r}(v)| + (C_5\alpha)\|\nabla v\|^2_{0,r} + (C_6\ \theta_\alpha + 1)\|v\|^2_{0,r}$$

where $\alpha > 0$ is arbitrary. Defining α by $C_5\alpha = 1/2$ and $C = C(p,\zeta,r,r')$ $= 2(C_5\theta_\alpha + 1)$ gives

$$(8.101) \qquad \|u\|^2_{1;0,r'} \le \frac{1}{2} C(|A_{p,\zeta,r}(v)| + \|v\|^2_{0,r}) + \frac{1}{2}\|\nabla v\|^2_{0,r}$$

since $\frac{1}{2}C = C_5\alpha + 1 \geq 1$. Finally, (8.101) implies (8.81) because $\|\nabla v\|_{0,r} \le \|u\|_{1;0,r'}$.

Proof of Theorem 8.2. The denseness of $D(A_{p,\zeta,r})$ in $L_2(\Omega_{0,r})$ follows from the obvious inclusion $C_0^\infty(\Omega_{0,r}) = Q_{p,\zeta,r}\, C_0^\infty(\Omega_{0,r}) \subset D(A_{p,\zeta,r})$. The sectorial property of $A_{p,\zeta,r}$ was proved as Corollary 8.5 above. To prove that $A_{p,\zeta,r}$ is closed let $v^{(n)} = Q_{p,\zeta,r}\, u^{(n)}$, with $u^{(n)} \in G_{p,\zeta,r}$, be $A_{p,\zeta,r}$-convergent to $v \in L_2(\Omega_{0,r})$; i.e., $v^{(n)} \to v$ in $L_2(\Omega_{0,r})$ and $A_{p,\zeta,r}(v^{(n)} - v^{(m)}) \to 0$ when $n,m \to \infty$. It must be shown that $v = Q_{p,\zeta,r}u$ where $u \in G_{p,\zeta,r}$ and $A_{p,\zeta,r}(v - v^{(n)}) \to 0$ when $n \to \infty$ [17, p. 313]. Now Theorem 8.6 applied to $v^{(n)} - v^{(m)} = Q_{p,\zeta,r}(u^{(n)} - u^{(m)})$ implies that $\{u^{(n)}\}$ is a Cauchy sequence in $G_{p,\zeta,r}$ and hence $\lim u^{(n)} = u \in G_{p,\zeta,r}$ exists. Clearly $v = Q_{p,\zeta,r}u$ since $Q_{p,\zeta,r}$ is bounded. Moreover, the convergence of $\{u^{(n)}\}$ to u in $G_{p,\zeta,r}$ implies that $\|\nabla v - \nabla v^{(n)}\|_{0,r} \to 0$ when $n \to \infty$. Hence, the representation (8.94) of $A_{p,\zeta,r}(v)$ implies that to complete the proof of Theorem 6.2 it will be enough to show that

$$(8.102) \qquad \lim_{n\to\infty} \sum_{m\in Z} w_{p+m}(\zeta)\ |u_m(r) - u_m^{(n)}(r)|^2 = 0.$$

Now Lemma 8.3 and the relation (8.60), applied to the partial sums of the Fourier series of $u \in G_{p,\zeta,r}$ in $\Omega_{r,r'}$, imply that

$$2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\ |u_m(r)|^2$$

(8.103)

$$= (u(\cdot,r'),D_y u(\cdot,r'))_{L_2(-\pi,\pi)} + \pi(\zeta)\|u\|^2_{r,r'} - \|\nabla u\|^2_{r,r'}.$$

It follows that (8.102) holds if

(8.104) $$\lim_{n\to\infty} (u(\cdot,r') - u^{(n)}(\cdot,r'),D_y u(\cdot,r') - D_y u^{(n)}(\cdot,r'))_{L_2(-\pi,\pi)} = 0.$$

To prove this define $s = 1/2\ (r+r')$, $s' = r + r'$ so that $r < s < r' < s'$. Then a Sobolev imbedding theorem [1] implies that

$$|(u(\cdot,r'),D_y u(\cdot,r'))| \le \|u(\cdot,r')\|\ \|D_y u(\cdot,r')\|$$

(8.105)

$$\le \|u\|_{1;s,r'}\ \|u\|_{2;s,r'}$$

$$\le \|u\|^2_{2;s,r'}.$$

Moreover, the interior elliptic estimates of [1] imply that there exists a $C = C(r,r')$ such that (see (8.42))

(8.106) $$\|u\|^2_{2;s,r'} \le C(\|u\|^2_{r,s'} + \|\Delta u\|^2_{r,s'}).$$

Since $\Delta u = -\pi(\zeta)u$ in Ω_r, (8.105) and (8.106) imply

(8.107) $$|(u(\cdot,r'),D_y u(\cdot,r'))| \le C'\ \|u\|^2_{r,s'}$$

where $C' = C'(p,\zeta,r,r')$. Applying (8.107) to $u - u^{(n)}$ gives (8.104). This completes the proof of Theorem 8.2. Note that the proof actually implies

Corollary 8.13. $Q_{p,\zeta,r}$ is a topological isomorphism of the Fréchet space $\mathcal{G}_{p,\zeta,r}$ onto $D(\mathcal{A}_{p,\zeta,r}) = Q_{p,\zeta,r}\,\mathcal{G}_{p,\zeta,r}$, topologized by the norm

$$(8.108)\qquad (|\mathcal{A}_{p,\zeta,r}(v)| + \|v\|^2_{0,r})^{1/2}\,.$$

Proof of Theorem 4.5. The densely defined, sectorial, closed sesquilinear form $\mathcal{A}_{p,\zeta,r}$ is associated with a unique m-sectorial operator $T_{p,\zeta,r}$ in $L_2(\Omega_{0,r})$ by Kato's first representation theorem [17, p. 322]. Theorem 4.5 will be proved by showing that $A_{p,\zeta,r} = T_{p,\zeta,r}$.

The Inclusion $A_{p,\zeta,r} \subset T_{p,\zeta,r}$. To prove this let $v \in D(A_{p,\zeta,r}) = P_{p,\zeta,r}\,\mathcal{F}_{p,\zeta,r} \subset D(\mathcal{A}_{p,\zeta,r})$ and write $z = -\Delta v = A_{p,\zeta,r}v$. It will be shown that

$$(8.109)\qquad \mathcal{A}_{p,\zeta,r}(v',v) = (v',A_{p,\zeta,r}v)_{0,r} = (v',z)_{0,r},\ v' \in D(\mathcal{A}_{p,\zeta,r}).$$

Note that this implies that $v \in D(T_{p,\zeta,r})$ and $T_{p,\zeta,r}v = z = A_{p,\zeta,r}v$, whence $A_{p,\zeta,r} \subset T_{p,\zeta,r}$.

Equation (8.109) will be proved by applying the generalized Dirichlet or Neumann boundary condition to v; i.e., the integral identities of the definitions (3.19), (3.20) of $D(A_p^{\ell oc})$. To this end let $r' > r$ and let $\phi_{r,r'}(y) \in C^\infty(R)$ be a cut-off function with the properties

$$(8.110)\qquad \phi_{r,r'}(y) = \begin{cases} 1, & y \le (2r + r')/3, \\ 0, & y \ge (r + 2r')/3, \end{cases}$$

and $\phi'_{r,r'}(y) \le 0$ (whence $0 \le \phi_{r,r'}(y) \le 1$). Then

$$v_{r,r'} = \phi_{r,r'}v' \in L_2^{D,p,com}(\Omega) \text{ or } L_2^{1,p,com}(\Omega) \tag{8.111}$$

and

$$\nabla v_{r,r'} = \phi_{r,r'}\, \nabla v' + \phi'_{r,r'}\, v'\hat{y} \tag{8.112}$$

where $\hat{y}$ is a unit vector in the y-direction. The integral identity of (3.19) or (3.20), applied to $v \in D(A_p^{\ell oc})$ and $v_{r,r'}$ gives

$$\begin{aligned} 0 &= (v_{r,r'},\Delta v)_{0,r'} + (\nabla v_{r,r'},\nabla v)_{0,r'} \\ &= (\phi_{r,r'}v',\Delta v)_{0,r'} + (\phi_{r,r'}\nabla v',\nabla v)_{0,r'} + (\phi'_{r,r'}v',D_y v)_{0,r'} \end{aligned} \tag{8.113}$$

Now the last term satisfies

$$\begin{aligned} (\phi'_{r,r'}v',D_y v)_{0,r'} &= \int_r^{r'} \int_{-\pi}^{\pi} \phi'_{r,r'}\, \overline{v'(x,y)}\, D_y v(x,y)\, dxdy \\ &= \int_r^{r'} \phi'_{r,r'}(y) \left(\int_{-\pi}^{\pi} \overline{v'(x,y)}\, D_y v(x,y)dx\right) dy \\ &\to -\int_{-\pi}^{\pi} \overline{v'(x,\gamma)}\, D_y v(x,\gamma)dx \\ &= -2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\, \overline{v'_m(r)}\, v_m(r),\ r' \to r; \end{aligned} \tag{8.114}$$

see [34, p. 57] for a similar calculation. Thus passage to the limit $r' \to r$ in (8.113) gives

$$(v',\Delta v)_{0,r} + (\nabla v',\nabla v)_{0,r} - 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\, \overline{v'_m(r)}\, v_m(r) = 0 \tag{8.115}$$

for all $v' \in D(\mathcal{A}_{p,\zeta,r})$. The definition (8.52) of $\mathcal{A}_{p,\zeta,r}$ implies that (8.115) is equivalent to (8.109).

<u>The Inclusion $T_{p,\zeta,r} \subset A_{p,\zeta,r}$</u>. To prove this let $v \in D(T_{p,\zeta,r})$ and $T_{p,\zeta,r}v = z \in L_2(\Omega_{0,r})$. This is equivalent to the identity

(8.116) $$\mathcal{A}_{p,\zeta,r}(v',v) = (v',z)_{0,r}$$

or

(8.117) $$(\nabla v',\nabla v)_{0,r} - 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\, \overline{v_m'(r)}\, v_m(r) = (v',z)_{0,r}$$

for all $v' \in D(\mathcal{A}_{p,\zeta,r})$. Taking $v' \in C_0^\infty(\Omega_{0,r})$ gives

(8.118) $$-\Delta v = z \text{ in } \Omega_{0,r}$$

by elementary distribution theory. Thus to complete the proof it is enough to show that $v \in D(A_{p,\zeta,r})$. Note that the definitions of $F_{p,\zeta,r}$ and $G_{p,\zeta,r}$ imply

(8.119) $$F_{p,\zeta,r} = G_{p,\zeta,r} \cap L_2^{\ell oc}(\Delta,\Omega).$$

Thus it will suffice to show that $u = Q_{p,\zeta,r}^{-1}v$ satisfies $\Delta u \in L_2^{\ell oc}(\Omega)$. This will be done by calculating the distribution Δu. To this end note that for all $\psi \in C_0^\infty(\Omega)$ one has

(8.120) $$(-\Delta\psi,u)_{L_2(\Omega)} = (\nabla\psi,\nabla u)_{L_2(\Omega)}$$

because $u \in L_2^{1,\ell oc}(\Omega)$. Thus

(8.121) $$(-\Delta\psi,u)_{L_2(\Omega)} = (\nabla\psi,\nabla v)_{0,r} + (\nabla\psi,\nabla u)_{r,\infty}.$$

Now equation (8.117) with $v' = \psi$ gives

$$(8.122)\qquad (\nabla\psi,\nabla v)_{0,r} = (\psi,z)_{0,r} + 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\ \overline{\psi_m(r)}\ v_m(r).$$

It will be shown that the last term in (8.121) satisfies

$$(8.123)\qquad (\nabla\psi,\nabla u)_{r,\infty} = \pi(\zeta)(\psi,u)_{r,\infty} - 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\ \overline{\psi_m(r)}\ v_m(r).$$

Adding equations (8.122) and (8.123) and using (8.121) gives

$$(8.124)\qquad (-\Delta\psi,u)_{L_2(\Omega)} = (\psi,z)_{0,r} + \pi(\zeta)(\psi,u)_{r,\infty} = (\psi,f)_{L_2(\Omega)}$$

where

$$(8.125)\qquad f(X) = \begin{cases} z(X), & X\in\Omega_{0,r}, \\ \pi(\zeta)\ u(X), & X\in\Omega_{r,\infty}. \end{cases}$$

Thus $-\Delta u = f \in L_2^{\ell oc}(\Omega)$.

The proof of Theorem 4.5 will be completed by verifying (8.123). To this end recall that $u \in G_{p,\zeta,r}$ and $\Delta u = -\pi(\zeta)u$ as a distribution in $\Omega_{r,\infty}$. Now define $\theta_{r,r'}(y) = 1 - \phi_{r,r'}(y)$, where $\phi_{r,r'}$ is defined as above, and define

$$(8.126)\qquad \psi_{r,r'} = \theta_{r,r'}\psi \in C_0^\infty(\Omega).$$

Then the distribution definitions of ∇u and Δu in $\Omega_{r,\infty}$ imply

$$(8.127)\qquad \begin{aligned}(\nabla\psi_{r,r'},\nabla u)_{r,\infty} &= (-\Delta\psi_{r,r'},u)_{r,\infty} = (\psi_{r,r'},-\Delta u)_{r,\infty} \\ &= \pi(\zeta)(\psi_{r,r'},u)_{r,\infty}.\end{aligned}$$

On the other hand, proceeding as in the first part of the proof one finds

$$(8.128)\qquad \begin{aligned}(\nabla\psi_{r,r'},\nabla u)_{r,\infty} &= (\theta_{r,r'}\nabla\psi,\nabla u)_{r,\infty} + (\theta'_{r,r'}\psi,D_y u)_{r,\infty}\\ &\to (\nabla\psi,\nabla u)_{r,\infty} + 2\pi i \sum_{m\in Z} w_{p+m}(\zeta)\,\overline{\psi_m(r)}\,u_m(r)\end{aligned}$$

when $r' \to r$. Thus passage to the limit $r' \to r$ in (8.127) gives (8.123) because $v = Q_{p,\zeta,r}u$ satisfies $v_m(r) = u_m(r)$.

<u>Proof of Theorem 4.6</u>. The proof of the continuity of $\{A_{p,\zeta,r} \mid (p,\zeta) \in M\}$ will be based on a criterion established by Kato [17, Theorem IV-2.29]. Thus for each $(p_0,\zeta_0) \in M$ one must construct a Hilbert space $\mathcal{H}$, a neighborhood $N(p_0,\zeta_0) \subset M$, operators $U(p,\zeta)$, $V(p,\zeta) \in B(\mathcal{H},L_2(\Omega_{0,r}))$ for $(p,\zeta) \in N(p_0,\zeta_0)$, and operators $U,V \in B(\mathcal{H},L_2(\Omega_{0,r}))$ with the properties that $U(p,\zeta)$ and U map $\mathcal{H}$ one-to-one onto $D(A_{p,\zeta,r})$ and $D(A_{p_0,\zeta_0,r})$, respectively,

$$(8.129)\qquad A_{p,\zeta,r}\,U(p,\zeta) = V(p,\zeta),\quad A_{p_0,\zeta_0,r}\,U = V,$$

and

$$(8.130)\qquad \|U(p,\zeta) - U\| \to 0,\ \|V(p,\zeta) - V\| \to 0 \text{ when } (p,\zeta) \to (p_0,\zeta_0).$$

The space $\mathcal{H}$ will be defined by

$$(8.131)\qquad \mathcal{H} = D(A_{p_0,\zeta_0,r}) \subset L_2^1(\Delta,\Omega_{0,r}).$$

Theorem 4.4 implies that $\mathcal{H}$ is closed in the topology of $L_2^1(\Delta,\Omega_{0,r})$ and hence is a Hilbert space. Next a neighborhood $N(p_0,\zeta_0)$ and linear operators

$$(8.132)\qquad J(p,\zeta,p_0,\zeta_0) \in B(\mathcal{H},L_2^1(\Delta,\Omega_{0,r})),\ (p,\zeta) \in N(p_0,\zeta_0),$$

will be constructed with the properties

(8.133) $\quad J(p,\zeta,p_0,\zeta_0)$ maps $\mathcal{H}$ one-to-one onto $D(A_{p,\zeta,r})$,

(8.134) $\quad J(p_0,\zeta_0,p_0,\zeta_0) = E$ is the natural embedding of $\mathcal{H}$ in $L_2^1(\Delta,\Omega_{0,r})$,

(8.135) $\quad (p,\zeta) \to J(p,\zeta,p_0,\zeta_0) \in B(\mathcal{H},L_2^1(\Delta,\Omega_{0,r}))$ is continuous at (p_0,ζ_0).

The desired operators can then be defined by

$$U(p,\zeta) = E_0\ J(p,\zeta,p_0,\zeta_0),\ U = U(p_0,\zeta_0), \tag{8.136}$$

$$V(p,\zeta) = A_{p,\zeta,r}\ U(p,\zeta), \quad V = V(p_0,\zeta_0), \tag{8.137}$$

where $E_0 : L_2^1(\Delta,\Omega_{0,r}) \to L_2(\Omega_{0,r})$ is the natural embedding. It is clear that these operators are in $B(\mathcal{H},L_2(\Omega_{0,r}))$ and $U(p,\zeta)$, U map $\mathcal{H}$ one-to-one onto $D(A_{p,\zeta,r})$, $D(A_{p_0,\zeta_0,r})$, respectively. Equations (8.129) hold by definition. Moreover,

$$\begin{aligned} \|U(p,\zeta) - U\| &= \|E_0(J(p,\zeta,p_0,\zeta_0) - J(p_0,\zeta_0,p_0,\zeta_0))\| \\ &\le \|J(p,\zeta,p_0,\zeta_0) - E\| \to 0 \end{aligned} \tag{8.138}$$

when $(p,\zeta) \to (p_0,\zeta_0)$ by (8.134), (8.135). Similarly, for all $u \in \mathcal{H}$,

$$\begin{aligned} \|(V(p,\zeta) - V)u\|_{0,r} &= \|\Delta J(p,\zeta,p_0,\zeta_0)u - \Delta J(p_0,\zeta_0,p_0,\zeta_0)u\|_{0,r} \\ &\le \|J(p,\zeta,p_0,\zeta_0)u - Eu\|_{1,\Delta;0,r} \\ &\le \|J(p,\zeta,p_0,\zeta_0) - E\|\ \|u\|_{\mathcal{H}} \end{aligned} \tag{8.139}$$

whence

$$(8.140) \qquad \|V(p,\zeta) - V\| \leq \|J(p,\zeta,p_0,\zeta_0) - E\| \to 0$$

when $(p,\zeta) \to (p_0,\zeta_0)$. The proof of Theorem 4.6 will be completed by constructing the family $J(p,\zeta,p_0,\zeta_0)$. The cases of the Dirichlet and Neumann boundary conditions will be treated separately.

<u>Construction of J - The Dirichlet Case</u>. The construction generalizes one of Alber [3]. To describe it let $v \in \mathcal{H} = D(A_{p_0,\zeta_0,r})$; i.e., $v = P_{p,\zeta,r}u$, $u \in F_{p_0,\zeta_0,r}$. The Fourier expansions of v and u have the forms

$$(8.141) \qquad v(x,y) = \sum_{m\in Z} v_m(y) \exp\{i(p_0 + m)x\}, \ (x,y) \in \Omega_{h,r},$$

$$(8.142) \qquad u(x,y) = \sum_{m\in Z} u_m(y) \exp\{i(p_0 + m)x\}, \ (x,y) \in \Omega_{h,\infty}.$$

Moreover, $v_m(y) = u_m(y)$ for $h \leq y \leq r$ and

$$(8.143) \qquad u_m(y) = c_m \exp\{i\, y\, w_{p_0+m}(\zeta_0)\} \text{ for } y \geq r.$$

Next introduce a function $\xi \in C^\infty(R)$ such that

$$(8.144) \qquad \xi(y) = \begin{cases} 1 \text{ for } -\infty < y \leq r_1 = (r + 2h)/3, \\ 0 \text{ for } r_2 = (2r + h)/3 \leq y < \infty \end{cases}$$

and $\xi'(y) \leq 0$ (whence $0 \leq \xi(y) \leq 1$), and define, for each $y \in R$,

$$(8.145) \qquad d_m(p,\zeta,p_0,\zeta_0,y) = \exp\{i\, y\, [w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)]\}[1 - \xi(y)] + \xi(y).$$

Choice of $N(p_0,\zeta_0)$. The equicontinuity of the functions $w_{p+m}(\zeta)$, Theorem 4.2, implies that there exists a neighborhood $N(p_0,\zeta_0) \subset M$ such that

$$|\exp\{i\,y\,[w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)]\} - 1| < 1/2 \tag{8.146}$$

for all $(p,\zeta) \in N(p_0,\zeta_0)$, $m \in Z$ and $y \in R$. Thus, using $||z_1| - |z_2|| \le |z_1 - z_2|$ one has

$$\begin{aligned} ||d_m(p,\zeta,p_0,\zeta_0,y)| - 1| &\le |d_m(p,\zeta,p_0,\zeta_0,y) - 1| \\ &\le |\exp\{i\,y\,[w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)]\} - 1|\,|1 - \xi(y)| < 1/2 \end{aligned} \tag{8.147}$$

and hence

$$1/2 < |d_m(p,\zeta,p_0,\zeta_0,y)| < 3/2 \tag{8.148}$$

for all $(p,\zeta) \in N(p_0,\zeta_0)$, $m \in Z$ and $y \in R$.

Definition. For all $v \in \mathcal{H} = D(A_{p_0,\zeta_0,r})$ with expansion (8.141) on $\Omega_{h,r}$ let

$$\begin{aligned} &J(p,\zeta,p_0,\zeta_0)v(x,y) \\ &= \exp\{i(p-p_0)x\} \begin{cases} \sum_{m\in Z} d_m(p,\zeta,p_0,\zeta_0,y)\, v_m(y) \exp\{i(p_0+m)x\} & \text{in } \Omega_{h,r}, \\ v(x,y) & \text{in } \Omega_{0,h}. \end{cases} \end{aligned} \tag{8.149}$$

Note that $d_m(p,\zeta,p_0,\zeta_0,y) \equiv 1$ and hence $J(p,\zeta,p_0,\zeta_0)\, v(x,y) = \exp\{i(p-p_0)x\}\, v(x,y)$ for $(x,y) \in \Omega_{h,r_1}$. Thus the definition produces no discontinuities at $y = h$. The proof that J has the properties (8.132) - (8.135) will be developed in several lemmas.

<u>Lemma 8.14</u>. There exists a constant $M = M(N(p_0,\zeta_0))$ such that

$$|D_y^k \, d_m(p,\zeta,p_0,\zeta_0,y)| \le M \tag{8.150}$$

for all $(p,\zeta) \in N(p_0,\zeta_0)$, $m \in Z$, $y \in R$ and $k = 0,1,2$.

This result follows easily from the definition (8.145) and the equicontinuity of the family $\{w_{p+m}(\zeta)\}$.

<u>Lemma 8.15</u>. J satisfies (8.132); i.e., for all $v \in D(A_{p_0,\zeta_0,r})$ one has $J(p,\zeta,p_0,\zeta_0)v \in L_2^1(\Delta,\Omega_{0,r})$ and there exists a $C = C(p_0,\zeta_0)$ such that

$$\|J(p,\zeta,p_0,\zeta_0)v\|_{1,\Delta;0,r} \le C \, \|v\|_{1,\Delta;0,r} \tag{8.151}$$

for all $v \in D(A_{p_0,\zeta_0,r})$ and all $(p,\zeta) \in N(p_0,\zeta_0)$.

<u>Proof of Lemma 8.15</u>. For all $v \in D(A_{p_0,\zeta_0,r})$ one has

$$\begin{aligned} \|v\|^2_{1,\Delta;0,r} &= \int_{\Omega_{0,r}} \{|v|^2 + |\nabla v|^2 + |\Delta v|^2\} \, dX \\ &= \|v\|^2_{1,\Delta;0,h} + \|v\|^2_{1,\Delta;h,r} \\ &= \|v\|^2_{1,\Delta;0,h} + 2\pi \sum_{m\in Z} \int_h^r I_m^0(y) \, dy \end{aligned} \tag{8.152}$$

where

$$I_m^0(y) = (1+|p_0+m|^2) \, |v_m|^2 + |D_y v_m|^2 + |D_y^2 v_m - (p_0+m)^2 v_m|^2. \tag{8.153}$$

Similarly, writing

$$J \, v_m(y) = d_m(p,\zeta,p_0,\zeta_0,y) \, v_m(y), \tag{8.154}$$

one has

$$\|J(p,\zeta,p_0,\zeta_0)v\|^2_{1,\Delta;0,r} = \|\exp\{i(p-p_0)\cdot\}v\|^2_{1,\Delta;0,h} + 2\pi \sum_{m\in Z} \int_h^r I_m(y)\, dy \tag{8.155}$$

where

$$I_m(y) = (1+|p+m|^2)|Jv_m|^2 + |D_y Jv_m|^2 + |D_y^2 Jv_m - (p+m)^2 Jv_m|^2. \tag{8.156}$$

Now a simple calculation gives the estimate

$$\|\exp\{i(p-p_0)\cdot\}v\|_{1,\Delta;0,h} \le C_1 \|v\|_{1,\Delta;0,h} \tag{8.157}$$

where $C_1 = C_1(N(p_0,\zeta_0))$. Similarly, Lemma 8.14 implies that there is a constant $C_2 = C_2(N(p_0,\zeta_0))$ such that

$$I_m(y) \le C_2^2\, I_m^0(y) \tag{8.158}$$

for all $(p,\zeta) \in N(p_0,\zeta_0)$, $m \in Z$ and $h \le y < \infty$. It follows that $J(p,\zeta,p_0,\zeta_0)v \in L_2^1(\Delta,\Omega_{0,r})$ and (8.151) holds with $C^2 = \text{Max}\,(C_1^2, 2\pi C_2^2)$.

<u>Lemma 8.16</u>. For all $v \in D(A_{p_0,\zeta_0,r})$ one has

$$J(p,\zeta,p_0,\zeta_0)v \in D(A_{p,\zeta,r}). \tag{8.159}$$

<u>Proof of Lemma 8.16</u>. Since $D(A_{p,\zeta,r}) = P_{p,\zeta,r}\, F_{p,\zeta,r}$ it must be shown that $\tilde v = J(p,\zeta,p_0,\zeta_0)v$ has a continuation $\tilde u$ to Ω which is in $F_{p,\zeta,r}$. Recall that for $h \le y \le r$ one has $u_m(y) = v_m(y)$ and hence

$$\tilde v(x,y) = \sum_{m\in Z} d_m(p,\zeta,p_0,\zeta_0,y)\, u_m(y) \exp\{i(p+m)x\} \tag{8.160}$$

where $u_m(y)$ is defined by (8.142), (8.143). Moreover, for $h < r_2 \le y \le r$ one has

(8.161) $$d_m\,(p,\zeta,p_0,\zeta_0,y) = \exp\,\{i\,y\,[w_{p+m}(\zeta) - w_{p_0+m}(\zeta_0)]\}$$

and hence it is natural to define the continuation of $\tilde{v}$ by (8.142), (8.143) and

(8.162) $$\tilde{u}(x,y) = \sum_{m\in Z} c_m \exp\,\{i\,x\,(p+m) + i\,y\,w_{p+m}(\zeta)\},\ y \geq r.$$

It is clear from the convergence of (8.142) in $L_2^{2,\ell oc}(\Omega_h)$ (Lemma 4.1) and Lemma 8.14 that (8.162) converges in $L_2^{2,\ell oc}(\Omega_r)$ and hence $\tilde{u} \in L_2^{1,\ell oc}(\Delta,\Omega)$. Also, the p_0-periodic boundary condition satisfied by v, together with (8.149) and (8.162), imply that $\tilde{u}$ satisfies the p-periodic boundary condition. Moreover, $\tilde{u}(x,y) = \exp\,\{i(p-p_0)x\}\,u(x,y)$ in $\Omega_{0,h}$ and hence $\tilde{u}$ satisfies the generalized Dirichlet condition (i.e., $\tilde{u} \in L_2^{D,p,\ell oc}(\Omega)$) because $u \in L_2^{D,p_0,\ell oc}(\Omega)$. *The preceding shows that* $\tilde{u} \in D(A_p^{D,\ell oc})$. Finally the expansion (8.162) has the form (4.28) corresponding to $(p,\zeta) \in M$ and hence $\tilde{u} \in F_{p,\zeta,r}$.

Lemma 8.17. $J(p,\zeta,p_0,\zeta_0)$ maps $D(A_{p_0,\zeta_0,r})$ one-to-one onto $D(A_{p,\zeta,r})$.

Proof of Lemma 8.17. Lemma 8.16 implies that $J(p,\zeta,p_0,\zeta_0)$ maps $D(A_{p_0,\zeta_0,r})$ into $D(A_{p,\zeta,r})$. Moreover, it is clear from (8.149) and (8.141) that $J(p,\zeta,p_0,\zeta_0)$ is injective. The surjectivity may be verified by constructing the inverse. To do this let $v = P_{p,\zeta,r}u \in D(A_{p,\zeta,r})$ and

(8.163) $$v(x,y) = \sum_{m\in Z} v_m(y)\,\exp\,\{i(p+m)x\}\ \text{in}\ \Omega_{h,r}$$

and define

$$v_0(x,y) = \exp\{i(p_0-p)x\} \begin{cases} \sum_{m\in Z} d_m(p,\zeta,p_0,\zeta_0,y)^{-1}\, v_m(y) \exp\{i(p+m)x\} & \text{in } \Omega_{h,r}, \\ v(x,y) & \text{in } \Omega_{0,h}. \end{cases} \tag{8.164}$$

Note that $|d_m(p,\zeta,p_0,\zeta_0,y)^{-1}| < 2$ for all $(p,\zeta) \in N(p_0,\zeta_0)$, $m \in Z$ and $y \in R$. Hence the technique used to prove Lemma 8.16 can be used to show that $v_0 \in D(A_{p_0,\zeta_0,r})$ and $J(p,\zeta,p_0,\zeta_0)v_0 = v$.

Property (8.134) is obvious from definition (8.149) because $d_m(p_0,\zeta_0,p_0,\zeta_0,y) \equiv 1$. Hence the verification of properties (8.132) - (8.135) of J may be completed by proving

Lemma 8.18. $(p,\zeta) \to J(p,\zeta,p_0,\zeta_0) \in B(\mathcal{H},L_2^1(\Delta,\Omega_{0,r}))$ is continuous at (p_0,ζ_0).

Proof of Lemma 8.18. It must be shown that $\|J(p,\zeta,p_0,\zeta_0) - E\| \to 0$ when $(p,\zeta) \to (p_0,\zeta_0)$. An equivalent condition is

$$\|J(p,\zeta,p_0,\zeta_0)v - Ev\|_{1,\Delta;0,r} \to 0 \text{ when } (p,\zeta) \to (p_0,\zeta_0), \tag{8.165}$$

uniformly for all $v \in \mathcal{H}$ such that $\|v\|_{1,\Delta;0,r} \le 1$ [17, p. 150]. To verify (8.165) define a bounded operator T_{p-p_0} in $L_2^1(\Delta,\Omega_{0,r})$ by

$$T_{p-p_0}\, v(x,y) = \exp\{i(p-p_0)x\}\, v(x,y). \tag{8.166}$$

Then for all $v \in \mathcal{H}$ one has

$$\begin{aligned} &\|J(p,\zeta,p_0,\zeta_0)v - Ev\|_{1,\Delta;0,r} \\ &\le \|J(p,\zeta,p_0,\zeta_0)v - T_{p-p_0}v\|_{1,\Delta;0,r} + \|T_{p-p_0}v - Ev\|_{1,\Delta;0,r}. \end{aligned} \tag{8.167}$$

Moreover one has, by (8.149) and (8.166),

$$J(p,\zeta,p_0,\zeta_0)\ v(x,y) - T_{p-p_0} v(x,y)$$

(8.168)
$$= \begin{cases} \sum_{m\in Z} \{d_m(p,\zeta,p_0,\zeta_0,y) - 1\}\ v_m(y)\ \exp\{i(p+m)x\} & \text{in } \Omega_{h,r}, \\ 0 & \text{in } \Omega_{0,h}, \end{cases}$$

whence

(8.169)
$$\| J(p,\zeta,p_0,\zeta_0)v - T_{p-p_0} v \|_{1,\Delta;0,r} = 2\pi \sum_{m\in Z} \int_h^r I_m^1(y)\ dy$$

where

(8.170)
$$I_m^1(y) = (1 + |p+m|^2)|f_m\ v_m|^2 + |(D_y\ f_m)v_m + f_m\ D_y\ v_m|^2$$
$$+ |f_m\ D_y^2\ v_m + 2\ D_y\ f_m\ D_y\ v_m + (D_y^2 f_m)v_m - (p+m)^2 f_m\ v_m|^2$$

and

(8.171)
$$f_m = f_m(p,\zeta,p_0,\zeta_0,y) = d_m(p,\zeta,p_0,\zeta_0,y) - 1.$$

Now using the equicontinuity of the family $\{w_{p+m}(\zeta)\}$ and Lemma 8.14 it is not difficult to show that for each $\varepsilon > 0$ there is a neighborhood $N'(p_0,\zeta_0)$ of (p_0,ζ_0) in M such that

(8.172)
$$0 \le I_m^1(y) \le \varepsilon^2\ I_m^0(y)$$

for all $(p,\zeta) \in N'(p_0,\zeta_0)$, $m \in Z$ and $h \le y \le r$, where $I_m^0(y)$ is defined by (8.153). It follows that (see (8.152))

$$\| J(p,\zeta,p_0,\zeta_0)v - T_{p-p_0} v \|_{1,\Delta;0,r} \leq \varepsilon \, \| v \|_{1,\Delta;0,r} \leq \varepsilon \tag{8.173}$$

for all $v \in \mathcal{H}$ such that $\| v \|_{1,\Delta;0,r} \leq 1$.

Similarly, an elementary calculation gives

$$\| T_{p-p_0} v - Ev \|_{1,\Delta;0,r} \leq \varepsilon \tag{8.174}$$

for all $v \in \mathcal{H}$ such that $\| v \|_{1,\Delta;0,r} \leq 1$. Combining (8.167), (8.173) and (8.174) gives (8.165).

<u>Construction of J - The Neumann Case</u>. The mapping J defined by (8.149) is not applicable to the Neumann case because the operation $v \to \exp\{i(p - p_0)x\}v$ does not preserve the Neumann boundary condition. It will be shown that for grating domains $G \in S$ a suitable mapping J can be defined by replacing the multiplier $\exp\{i(p - p_0)x\}$ by a function of the form $\exp\{i(p - p_0)\,\phi(x,y)\}$. To this end note that if x_0 has property (1.9) of the definition of the class S then so do the points $x_0 + 2\pi m$, $m \in Z$. Moreover, it can be assumed that $x_0 = -\pi$ since equivalent domains are obtained by translating G parallel to the x-axis. This assumption is made in the remainder of this section. Also, to simplify the notation it will be assumed that

$$\partial G \cap \{(-\pi,y) \mid y \in R\} = (-\pi,y_0) \tag{8.175}$$

is a single point. The general case defined by (1.9) can be treated by the same method.

Property S implies that near $(-\pi,y_0)$ the boundary Γ has a representation $(x,y) = (f_1(s),f_2(s))$, where s is the arc length on Γ measured from $(-\pi,y_0)$, and $f_j \in C^3$. The vectors $\vec{t} = (f_1'(s),f_2'(s))$ and

$\vec{n} = (-f_2'(s), f_1'(s))$ are unit tangent and normal vectors to Γ, respectively. The mapping $(s,t) \to (x,y)$ defined by

$$\text{(8.176)} \qquad \begin{aligned} x &= f_1(s) - t\, f_2'(s), \\ y &= f_2(s) + t\, f_1'(s), \end{aligned}$$

has Jacobian 1 at $(s,t) = (0,0)$. Hence the inverse mapping

$$\text{(8.177)} \qquad \begin{aligned} s &= \sigma(x,y), \\ t &= \tau(x,y), \end{aligned}$$

exists in a neighborhood of $(-\pi, y_0)$ and defines there a coordinate system of class C^2. The system is valid in a domain $\mathcal{O} = \{(s,t) : |s| < \delta_1, |t| < \delta_2\}$. It will be assumed that δ_1, δ_2 are chosen so small that $\mathcal{O} \subset \{(x,y) : |x + \pi| < \pi\}$. If extensions of $\sigma(x,y)$, $\tau(x,y)$ to $\mathcal{O} + (2\pi m, 0)$ are defined by $\sigma(x + 2\pi m, y) = \sigma(x,y)$ and $\tau(x + 2\pi, y) = \tau(x,y)$ then the extended functions define coordinate systems in $\mathcal{O} + (2\pi m, 0)$.

Introduce functions $\xi_j \in C(R)$ $(j = 1,2)$ such that $\xi_j(-\alpha) = \xi_j(\alpha)$, $\xi_j'(\alpha) \leq 0$ for $\alpha \geq 0$ and

$$\text{(8.178)} \qquad \xi_j(\alpha) = \begin{cases} 1, & |\alpha| \leq \delta_j/3, \\ 0, & |\alpha| \geq 2\delta_j/3, \end{cases}$$

(whence $0 \leq \xi_j(\alpha) \leq 1$). The composite functions $\xi_1(\sigma(x,y))$ and $\xi_2(\tau(x,y))$ are then in class C^2. Similarly, introduce a function $\xi_3(x)$ such that

$$\xi_3(x) = \begin{cases} 1, & |x + \pi| \le \delta_3/3, \\ 0, & 2\delta_3/3 \le |x + \pi| \le \delta_3, \end{cases} \tag{8.179}$$

and

$$\xi_3(x + 2\pi) = \xi_3(x) \tag{8.180}$$

where $\delta_3 < \pi$. Finally define

$$\begin{aligned} \phi(x,y) &= (\sigma-\pi)\ \xi_1(\sigma)\ \xi_2(\tau) + x\ \xi_3(x)[1 - \xi_2(\tau)],\ -\pi \le x \le 0, \\ \phi(x,y) &= (\sigma+\pi)\ \xi_1(\sigma)\ \xi_2(\tau) + x\ \xi_3(x)[1 - \xi_2(\tau)],\ 0 \le x \le \pi. \end{aligned} \tag{8.181}$$

The two parts of the definition are consistent because both give zero in a neighborhood of the y-axis. It will also be assumed that δ_2 is so small that $\xi_1(\sigma(\pm\pi,y)) = 1$ on the support of $\xi_2(\tau(\pm\pi,y))$.

The mapping J defined by (8.149) with $\exp\{i(p - p_0)x\}$ replaced by $\exp\{i(p - p_0)\ \phi(x,y)\}$ has the required properties (8.132) - (8.135). The proofs are the same as in the Dirichlet case except for the verification that $v' = J(p,\zeta,p_0,\zeta_0)v$ satisfies the Neumann and p-periodic boundary conditions. To verify the Neumann condition note that on the portion of Γ in the neighborhood defined by
$\operatorname{supp} \phi \cap \{(x,y) : |\tau(x,y)| \le \delta_2/3\}$ one has

$$\phi(x,y) = (\sigma(x,y) \pm \pi)\ \xi_1(\sigma(x,y)). \tag{8.182}$$

Moreover, on the regular portion of Γ a simple calculation based on (8.176) gives

(8.183) $$\sigma_x = \tau_y = f_1'(\sigma),\ \sigma_y = -\tau_x = f_2'(\sigma)$$

whence

(8.184) $$D_\nu\, \sigma = (-f_2'(\sigma))\sigma_x + (f_1'(\sigma))\sigma_y = 0.$$

It follows from (8.182) and (8.184) that $v'(x,y)$ ($= \exp\{i(p-p_0)\phi(x,y)\}v(x,y)$ on Ω_h) satisfies

(8.185) $$D_\nu\, v' = \exp\{i(p-p_0)\phi\}(D_\nu v + i(p-p_0)D_\nu\, \phi) = 0$$

on supp $\phi \cap \Gamma$. On the remainder of Γ $v' = v$ satisfies the generalized Neumann condition. The validity of the generalized Neumann condition for v' follows by a partition of unity argument.

To verify that v' satisfies the p-periodic boundary condition note that (8.181) and the assumption that $\xi_1(\sigma(\pm\pi,y)) = 1$ on the support of $\xi_2(\tau(\pm\pi,y))$ imply

(8.186)
$$\begin{aligned}
\phi(\pi,y) &= (\sigma(\pi,y) + \pi)\ \xi_2(\tau(\pi,y)) + \pi(1 - \xi_2(\tau(\pi,y))) \\
&= (\sigma(-\pi,y) + \pi)\ \xi_2(\tau(-\pi,y)) + \pi(1 - \xi_2(\tau(-\pi,y))) \\
&= \phi(-\pi,y) + 2\pi\ \xi_2(\tau(-\pi,y)) + 2\pi(1 - \xi_2(\tau(-\pi,y))) \\
&= \phi(-\pi,y) + 2\pi
\end{aligned}$$

and similarly

(8.187) $$D_x\, \phi(\pi,y) = D_x\, \phi(-\pi,y).$$

Thus

$$v'(\pi,y) = \exp\{i(p - p_0)\ \phi(\pi,y)\}\ v(\pi,y)$$

(8.188)

$$= \exp\{i(p-p_0)\ \phi(-\pi,y) + i(p-p_0)2\pi + 2\pi i p_0\}\ v(-\pi,y)$$

$$= \exp\{2\pi ip\}\ v'(-\pi,y)$$

and similarly

(8.189) $$D_x\ v'(x,y) = \exp\{i(p-p_0)\phi\}(D_x v + i(p-p_0)(D_x\phi)v)$$

whence

$$D_x v'(\pi,y) = \exp\{i(p-p_0)\ \phi(-\pi,y) + i(p-p_0)2\pi + 2\pi i p_0\} \times$$

(8.190)

$$\times\ \{D_x\ v(-\pi,y) + i(p-p_0)\ D_x\ \phi(-\pi,y)\ v(-\pi,y)\}$$

$$= \exp\{2\pi ip\}\ D_x\ v'(-\pi,y)$$

The above discussion completes the proof of the continuity of the family $\{A_{p,\zeta,r} : (p,\zeta) \in M\}$. The final assertion of Theorem 4.6 states that for fixed $p \in (-1/2,1/2]$ the family $\{A_{p,\zeta,r} : \zeta \in M_p\}$ is holomorphic in the generalized sense of Kato [17, p. 366]. This may be proved by means of the family of operators $J_p(\zeta,\zeta_0) \equiv J(p,\zeta,p,\zeta_0)$. It is only necessary to verify that $\zeta \to J_p(\zeta,\zeta_0)$ is holomorphic on M_p. A proof has been given by Alber [3, p. 271].

<u>Proof of Theorem 4.7</u>. $D(A_{p,\zeta,r}) = P_{p,\zeta,r}\ F_{p,\zeta,r}$ is a closed subspace of the Hilbert space $L_2^1(\Delta,\Omega_{0,r})$, by Theorem 4.4. $A_{p,\zeta,r} - z$ defines a bounded operator from this space into $L_2(\Omega_{0,r})$. Thus the operator $T : L_2(\Omega_{0,r}) \to D(A_{p,\zeta,r})$ defined by $Tf = R(A_{p,\zeta,r},z)f$ for all $f \in L_2(\Omega_{0,r})$ is closed and defined on all of $L_2(\Omega_{0,r})$. Thus T is bounded, by the closed graph theorem.

Next note that $R(A_{p,\zeta,r},z) = ET$ where $E : D(A_{p,\zeta,r}) \to L_2(\Omega_{0,r})$ is the natural embedding. Hence, the compactness of the resolvent of $A_{p,\zeta,r}$ follows from the compactness of E. Now, in the Neumann case $F_{p,\zeta,r} \subset L_2^{1,p,\ell oc}(\Omega)$ and hence the compactness of E follows from the hypothesis $G \in LC$. In the Dirichlet case, $F_{p,\zeta,r} \subset L_2^{D,p,\ell oc}(\Omega)$ = closure of $C_p^\infty(\Omega)$ in $L_2^{1,\ell oc}(\Omega)$. The last set can be regarded as a subset of $L_2^{1,p,\ell oc}(B_0)$ for which the natural embedding into $L_2^{\ell oc}(B_0)$ has the local compactness property. Hence, in this case E is compact without local restrictions on $\Gamma = \partial G \cap \Omega$. This proves the compactness of the resolvent of $A_{p,\zeta,r}$. The discreteness of $\sigma(A_{p,\zeta,r})$ follows immediately; see Kato [17, p. 187].

Proof of Theorem 4.8. It will be shown that if $\zeta \in M_p^+$ then the operator in $L_2(\Omega_{0,r})$ defined by

$$T = P_{p,\zeta,r}\, R(A_p,\pi_p(\zeta))\, P_r \tag{8.191}$$

is a bounded inverse of $A_{p,\zeta,r} - \pi_p(\zeta)$ in $L_2(\Omega_{0,r})$. To prove that T is a right inverse of $A_{p,\zeta,r} - \pi_p(\zeta)$ let $f \in L_2(\Omega_{0,r})$ and define $u = R(A_p,\pi_p(\zeta))f$. Then $u \in R(A_p)$ and

$$(A_p - \pi_p(\zeta))u = P_r f = \begin{cases} f \text{ in } \Omega_{0,r}, \\ 0 \text{ in } \Omega_r. \end{cases} \tag{8.192}$$

In particular, $(\Delta + \pi_p(\zeta))u = 0$ in Ω_r and thus since $u \in L_2(\Omega)$ the Fourier expansion (4.28) must hold with $\operatorname{Im} w_{p+m}(\zeta) > 0$ for all $m \in Z$. Thus $u \in F_{p,\zeta,r}$ and it follows that $P_{p,\zeta,r}u \in D(A_{p,\zeta,r})$ and

$$[A_{p,\zeta,r} - \pi_p(\zeta)]Tf = [A_{p,\zeta,r} - \pi_p(\zeta)]P_{p,\zeta,r}u = (-\Delta - \pi_p(\zeta))u\Big|_{\Omega_{0,r}} = f \tag{8.193}$$

by (8.192).

To prove that T is a left inverse of $A_{p,\zeta,r} - \pi_p(\zeta)$ let $v \in D(A_{p,\zeta,r})$. Then $u = P^{-1}_{p,\zeta,r}v$ has Fourier expansion (4.28) with $\operatorname{Im} w_{p+m}(\zeta) > 0$ for all $m \in Z$ because $\zeta \in M^+_p$. Thus $u \in F_{p,\zeta,r} \cap L^1_2(\Delta,\Omega) = D(A_p)$ and one has

$$
\begin{aligned}
P_r[A_{p,\zeta,r} - \pi_p(\zeta)]v &= (-\Delta - \pi_p(\zeta))P^{-1}_{p,\zeta,r}v \\
&= (A_p - \pi_p(\zeta))P^{-1}_{p,\zeta,r}v
\end{aligned}
\tag{8.194}
$$

whence $T[A_{p,\zeta,r} - \pi_p(\zeta)]v = v$.

Proof of Theorem 4.9. The family of operators $\{A_{p,\zeta,r} - \pi_p(\zeta) \mid \zeta \in M_p\}$ is holomorphic (Theorem 4.6) and has compact resolvents (Theorem 4.7). It follows from a theorem of Kato [17, p. 371] that either $\Sigma_p = M_p$ or Σ_p has no accumulation points in M_p. But $M^+_p \cap \Sigma_p = \phi$ by Theorem 4.8. Hence the second alternative must hold.

To prove that Σ_p is independent of $r > h$ let $h < r' < r$ and suppose that $\pi_p(\zeta) \in \sigma(A_{p,\zeta,r})$. Then there exists a non-zero $v \in D(A_{p,\zeta,r})$ such that $(A_{p,\zeta,r} - \pi_p(\zeta))v = 0$ in $L_2(\Omega_{0,r})$. But then $u = P^{-1}_{p,\zeta,r}u \in F_{p,\zeta,r} \subset L^{1,\ell oc}_2(\Delta,\Omega)$ and $(\Delta + \pi_p(\zeta))u = 0$ in all of Ω. In particular, the Fourier expansion (4.28) holds in $\Omega_{r',\infty}$. Thus $u \in F_{p,\zeta,r'}$ and hence $P_{p,\zeta,r'}u \in D(A_{p,\zeta,r'})$ and $(A_{p,\zeta,r'} - \pi_p(\zeta))P_{p,\zeta,r'}u = 0$. Thus $\pi_p(\zeta) \in \sigma(A_{p,\zeta,r'})$ as was to be shown. The same argument is applicable if $r' > r$.

Proof of Corollary 4.10. Theorem 4.7 implies that every $z \in C$ is either an eigenvalue of $A_{p,\zeta,r}$ or lies in $\rho(A_{p,\zeta,r})$. Hence for each $\zeta \in M_p - \Sigma_p$ one has $\pi_p(\zeta) \in \rho(A_{p,\zeta,r})$ and it follows from [17, p. 367] that $R_{p,\zeta,r}$ is holomorphic on $M_p - \Sigma_p$. Thus to complete the proof it is enough to show that each $\zeta_0 \in \Sigma_p$ is a pole of $R_{p,\zeta,r}$. This will be deduced from S. Steinberg's theorem [28] and the following

<u>Lemma 8.19</u>. Let $\zeta \in M_p^+$ and $\operatorname{Im} \pi_p(\zeta) > 0$ (resp., < 0). Then every $z \in \sigma(A_{p,\zeta,r})$ satisfies $\operatorname{Im} z < 0$ (resp. > 0).

<u>Proof of Lemma 8.19</u>. Let $v \in D(A_{p,\zeta,r})$ be an eigenfunction of $A_{p,\zeta,r}$ with eigenvalue z : $v \neq 0$ and $A_{p,\zeta,r}v = zv$. Then $u = P^{-1}_{p,\zeta,r}v \in F_{p,\zeta,r}$ and hence

$$(\Delta + z)u = (\Delta + z)v = 0 \text{ in } \Omega_{0,r}, \text{ and} \tag{8.195}$$

$$(\Delta + z')u = 0 \text{ in } \Omega_{r,\infty} \tag{8.196}$$

where $z' = \pi_p(\zeta)$. Moreover, Lemma 4.1 and Sobolev's embedding theorems imply that $y \to u(\cdot,y)$ is in $C^1([h,\infty),L_2(-\pi,\pi))$. In addition, the assumption $\zeta \in M_p^+$ implies that $u \in D(A_p) \subset L_2(\Omega)$.

Application of Green's theorem to u and $\bar{u}$ in $\Omega_{r,\infty}$ gives, by (8.196),

$$(-2i \operatorname{Im} z') \int_{\Omega_{r,\infty}} |u|^2 \, dX = - \int_{-\pi}^{\pi} \left\{ \bar{u} \frac{\partial u}{\partial y} - u \frac{\partial \bar{u}}{\partial y} \right\}_{y=r} dx. \tag{8.197}$$

Similarly, application of Green's theorem in $\Omega_{0,r}$ gives

$$(-2i \operatorname{Im} z) \int_{\Omega_{0,r}} |u|^2 \, dX = \int_{-\pi}^{\pi} \left\{ \bar{u} \frac{\partial u}{\partial y} - u \frac{\partial \bar{u}}{\partial y} \right\}_{y=r} dx. \tag{8.198}$$

Adding (8.197) and (8.198) gives

$$\operatorname{Im} z' \int_{\Omega_{r,\infty}} |u|^2 \, dX + \operatorname{Im} z \int_{\Omega_{0,r}} |u|^2 dX = 0. \tag{8.199}$$

Thus if $\operatorname{Im} \pi_p(\zeta) = \operatorname{Im} z' > 0$ and $\operatorname{Im} z \geq 0$ then $u(X) \equiv 0$ in $\Omega_{r,\infty}$. But then $u(\cdot,r) = 0$ and $D_y\, u(\cdot,r) = 0$ and hence $u(X) \equiv 0$ in $\Omega_{0,r}$ by the unique continuation property for (8.195). Hence $\operatorname{Im} \pi_p(\zeta) > 0$ implies $\operatorname{Im} z < 0$. The other case is proved in the same way.

Returning to the proof of Corollary 4.10, it will be shown first that every $\zeta_0 \in \Sigma_p$ such that

$$\operatorname{Im} \pi_p(\zeta_0) \geq 0 \tag{8.200}$$

is a pole of $R_{p,\zeta,r}$. To this end choose $\zeta_1 \in M_p^+$ such that $\operatorname{Im} \pi_p(\zeta_1) > 0$, so that

$$\{z \mid \operatorname{Im} z \geq 0\} \subset \rho(A_{p,\zeta_1,r}) \tag{8.201}$$

by Lemma 8.19. Next choose a $z_1 \in C$ such that

$$z_1 \in \rho(A_{p,\zeta_1,r}), \tag{8.202}$$

$$z_1 \in \rho(A_{p,\zeta,r}) \text{ for all } \zeta \in N(\zeta_0,\delta), \tag{8.203}$$

where $N(\zeta_0,\delta)$ is the component of $\pi_p^{-1}(D(\pi_p(\zeta_0),\delta))$ containing ζ_0. $N(\zeta_0,\delta)$ has compact closure and hence such numbers z_1 exist by Corollary 8.5 above. In the remainder of the proof the following notation is used:

$$\begin{cases} R(\zeta,z) = (A_{p,\zeta,r} - z)^{-1}, \\ R(\zeta) = R(\zeta,\pi_p(\zeta)). \end{cases} \tag{8.204}$$

With the above choices of ζ_1 and z_1 the operator

$$B(z) = (1 - (z - z_1)\, R(\zeta_1,z_1))^{-1} \tag{8.205}$$

exists and is holomorphic for $\operatorname{Im} z \geq 0$ (i.e., in an open set containing $\operatorname{Im} z \geq 0$). Indeed,

$$1 - (z - z_1)\, R(\zeta_1, z_1) = (A_{p,\zeta_1,r} - z)\, R(\zeta_1, z_1) \tag{8.206}$$

and the existence of $B(z)$ follows from (8.201). The analyticity follows from that of $R(\zeta_1, z)$.

To complete the proof of Corollary 4.10 note that (8.200) and (8.201) imply that $\pi_p(\zeta_0) \in \rho(A_{p,\zeta_1,r})$. Since the resolvent set is open, the continuity of π_p implies that there exists a $\delta > 0$ such that $\pi_p(\zeta) \in \rho(A_{p,\zeta_1,r})$ for all $\zeta \in N(\zeta_0, \delta)$. Hence $B(\pi_p(\zeta))$ exists and is holomorphic in $N(\zeta_0, \delta)$. Now for all such ζ one has, by (8.202), (8.203),

$$\begin{aligned} 1 - (\pi_p(\zeta) - z_1)\, R(\zeta, z_1) &= 1 - (\pi_p(\zeta) - z_1)\, R(\zeta_1, z_1) \\ &\quad - (\pi_p(\zeta) - z_1)\, \{R(\zeta, z_1) - R(\zeta_1, z_1)\}. \end{aligned} \tag{8.207}$$

Multiplying by $B(\pi_p(\zeta))$ gives

$$B(\pi_p(\zeta))\{1 - (\pi_p(\zeta) - z_1)\, R(\zeta, z_1)\} = 1 - T(\zeta) \tag{8.208}$$

where

$$T(\zeta) = (\pi_p(\zeta) - z_1)\, B(\pi_p(\zeta))\, \{R(\zeta, z_1) - R(\zeta_1, z_1)\} \tag{8.209}$$

defines a compact operator-valued holomorphic family in $N(\zeta_0, \delta)$. By Steinberg's theorem [28], $(1 - T(\zeta))^{-1}$ either exists nowhere or is meromorphic in $N(\zeta_0, \delta)$. The second case must hold because the singularities of $(1 - T(\zeta))^{-1}$ are those of $R(\zeta)$ and hence are isolated. In particular, for δ small enough $(1 - T(\zeta))^{-1}$ is analytic in $N(\zeta_0, \delta)$

except for a pole at $\zeta = \zeta_0$. Equation (8.208) then implies

$$A_{p,\zeta,r} - \pi_p(\zeta) = B(\pi_p(\zeta))^{-1}(1 - T(\zeta))\, R(\zeta,z_1)^{-1} \tag{8.210}$$

and therefore

$$R(\zeta) = R(\zeta,z_1)(1 - T(\zeta))^{-1}\, B(\pi_p(\zeta)) \tag{8.211}$$

for $\zeta \in N(\zeta_0,\delta) - \{\zeta_0\}$. This exhibits $R(\zeta)$ as a product of operators that are holomorphic at ζ_0 and one that has a pole there. The residue of $R(\zeta)$ at ζ_0 has finite rank [28] and hence $\pi_p(\zeta_0)$ is an eigenvalue of finite algebraic multiplicity [17, p. 181].

Proof of Corollary 4.11. This result follows immediately from Theorems 4.4 and 4.8.

Proof of Corollary 4.12. It will be shown that

$$\sigma_0(A_p) \subset \pi_p(\overline{M_p^+} \cap \Sigma_p). \tag{8.212}$$

The discreteness of $\sigma_0(A_p)$ will then follow from Theorem 4.9. To prove (8.212) let $\lambda \in \sigma_0(A_p) \subset \sigma(A_p) = [p^2,\infty)$ and let $\lambda \pm i0$ denote the points of $\overline{M_p^+}$ above λ so

$$\pi_p(\lambda \pm i0) = \lambda. \tag{8.213}$$

If $u \in D(A_p)$ is a corresponding eigenfunction of A_p then $u \in F_{p,\lambda\pm i0,r}$, $v_\pm = P_{p,\lambda\pm i0,r}u \in D(A_{p,\lambda\pm i0,r})$ and $(A_{p,\lambda\pm i0,r} - \lambda)v_\pm = 0$. Thus $(A_{p,\lambda\pm i0,r} - \pi_p(\lambda \pm i0))$ is not invertible and hence $\lambda \pm i0 \in \Sigma_p$.

The inclusion (8.212) and Theorem 4.9 imply that $\sigma_0(A_p)$ has no finite limit points. To show that each $\lambda \in \sigma_0(A_p)$ has a finite dimensional eigenspace note that the algebraic and geometric eigenspaces of A_p coincide because A_p is selfadjoint. Moreover $P_{p,\lambda\pm i0,r}$ maps the eigenspace of $\lambda \in \sigma_0(A_p)$ onto the geometric eigenspace of $A_{p,\lambda\pm i0,r}$ for λ, as was shown above. However, the latter coincides with the geometric eigenspace of the compact operator $R(\lambda \pm i0,z)$ defined by (8.204) and hence is finite dimensional.

<u>Proof of Corollary 4.13</u>. To prove (4.43) note that if $\lambda \in \pi_p(\overline{M_p^+} \cap \Sigma_p) - T_p$ then $\lambda + i0$ or $\lambda - i0$ is in $\overline{M_p^+} \cap \Sigma_p$ and hence $\lambda = \pi_p(\lambda \pm i0)$ is an eigenvalue of $A_{p,\lambda+i0,r}$ or $A_{p,\lambda-i0,r}$ with eigenfunction v_+ or v_-. But then $u_+ = P^{-1}_{p,\lambda+i0,r}\, v_+$ or $u_- = P^{-1}_{p,\lambda-i0,r}\, v_-$ will have a p-periodic extension to G that is a pure outgoing or incoming R-B wave for A. It follows from Theorem 2.1 that u_+ or u_- is an eigenfunction for A_p with eigenvalue λ; i.e., $\lambda \in \sigma_0(A_p)$.

<u>Proof of Theorem 4.14</u>. Both statements of Theorem 4.14 follow from the continuity of the family $\{A_{p,\zeta,r} : (p,\zeta) \in M\}$ and a theorem of Kato [17, Theorem IV.2.25]. Indeed, if $(p_0,\zeta_0) \in M - \Sigma$ then $\pi_{p_0}(\zeta_0) \in \rho(A_{p_0,\zeta_0,r})$ and hence $R_{p,\zeta,r} \to R_{p_0,\zeta_0,r}$ when $(p,\zeta) \to (p_0,\zeta_0)$. Moreover, it follows from Kato's theorem that there exists a neighborhood $N(p_0,\zeta_0,\rho,\delta) \subset M - \Sigma$.

<u>Proof of Theorem 4.15</u>. This result is an immediate corollary of Theorem 4.14 and Theorem 4.4.

<u>Proof of Corollary 4.16</u>. Theorem 4.15 implies that $(p,\zeta) \to P^{-1}_{p,\zeta,r}\, R_{p,\zeta,r} \in B(L_2(\Omega_{0,r}),\, L_2^1(\Delta,\Omega_{0,r'}))$ is continuous on $M - \Sigma$ for each $r' > r$. This implies (4.49) with

$$(8.214) \qquad C(K,r,r') = \operatorname*{Max}_{(p,\zeta)\in K} \| P^{-1}_{p,\zeta,r}\, R_{p,\zeta,r} \|_{r,r'}$$

where $\|\cdot\|_{r,r'}$ denotes the operator norm in the space $B(L_2(\Omega_{0,r}),L_2^1(\Delta,\Omega_{0,r'}))$.

<u>Proof of Corollary 4.17</u>. This result is a special case of Corollary 4.16.

§9. Proofs of the Results of §5.

Theorem 5.1 is a direct consequence of Theorem 2.1 and the results of §4.

Proof of Lemma 5.2. The proof follows the plan of [34, Lemma 6.3]. Definitions (5.16) and (5.25) imply that if $f \in L_2^{com}(\Omega)$ then

$$\tilde{f}(p+m,q,z) = \int_{\text{supp } f} \overline{\phi_0(X,p+m,q)}\, j(y)\, f(X)\, dX + \int_{\text{supp } f} \overline{\phi'(X,p+m,q,\bar{z})}\, f(X) dX$$

(9.1)

$$= (J f)_0^{\sim} (p+m,q) + \int_{\text{supp } f} \overline{R(A_p,\bar{z})\, M(\cdot,p+m,q)}\, f(X)\, dX$$

$$= (J f)_0^{\sim} (p+m,q) + \int_{\Omega_{h,r}} \overline{M(X,p+m,q)}\, R(A_p,z)\, f(X)\, dX$$

$$= (J f)_0^{\sim} (p+m,q) +$$

$$+ \int_{\Omega_{h,r}} \overline{(\Delta+\omega^2(p+m,q))\{j(y)\phi_0(X,p+m,q)\}}\, R(A_p,z)\, f(X)\, dX$$

since $\phi'(\cdot,p+m,q,z) = R(A_p,z)\, M(\cdot,p+m,q)$ by (5.13) and Theorem 4.8 and supp $M \subset \Omega_{h,r}$. The next-to-last equation follows from $R(A_p,z) = R(A_p,\bar{z})^*$. To derive (5.26) from (9.1) it is necessary to integrate by parts in the last integral. This cannot be done directly because $j(y)\, \phi_0(X,p+m,q) \notin L_2(\Omega)$. To complete the calculation introduce a function $\xi \in C^\infty(R)$ such that $\xi'(y) \leq 0$, $\xi(y) = 1$ for $y \leq 0$, $\xi(y) = 0$ for $y \geq 1$ and define

$$\xi_n(y) = \xi(y-n) = \begin{cases} 1, & y \le n, \\ 0, & y \ge n+1. \end{cases} \tag{9.2}$$

Then for $n \ge r$ one has $\xi_n(y) \equiv 1$ on $\Omega_{0,r}$ and hence

$$\begin{aligned} \tilde{f}(p+m,q,z) &= (J\,f)\tilde{_0}\,(p+m,q) \\ &+ \int_\Omega \overline{(\Delta+\omega^2(p+m,q))\{j(y)\phi_0(X,p+m,q)\}}\,\xi_n(y)\; R(A_p,z)\; f(X)\; dX \end{aligned} \tag{9.3}$$

Now

$$j(y)\;\phi_0(X,p+m,q) \in D(A_p^{N,\ell oc}(\Omega))\;\;(\text{resp. } D(A_p^{D,\ell oc}(\Omega)). \tag{9.4}$$

This may be shown by interpreting $\exp\{-ipx\}\;\phi_0(X,p+m,q)$ as a function on the cylinder Ω^Y (see the proof of Lemma 4.1) and recalling that $j(y) = 0$ for $0 \le y \le (h+r)/2$. Moreover,

$$\xi_n\; R(A_p,z)f \in L_2^{1,p,com}(\Omega)\;\;(\text{resp. } L_2^{D,p,com}(\Omega)) \tag{9.5}$$

since $R(A_p,z)f \in D(A_p)$. Conditions (9.4), (9.5) and the integral identities of (3.19), (3.20) applied to $u = j\phi_0$ and $v = \xi_n R(A_p,z)f$ give

$$\begin{aligned} &\int_\Omega \Delta\;\overline{\{j(y)\phi_0(X,p+m,q)\}}\;\xi_n(y)R(A_p,z)f(X)dX \\ &= -\int_\Omega \nabla\;\overline{\{j(y)\phi_0(X,p+m,q)\}}\cdot\nabla\{\xi_n(y)R(A_p,z)f(X)\}dX \end{aligned} \tag{9.6}$$

$$\text{(9.6 cont.)} \quad = -\int_{\Omega} \nabla\, \overline{\{j(y)\xi_{n+1}(y)\phi_0(X,p+m,q)\}} \cdot \nabla\{\xi_n(y)\tilde{j}(y)R(A_p,z)f(X)\}dX$$

where $\tilde{j} \in C_0^{\infty}(h,\infty)$ and $\tilde{j}(y) \equiv 1$ for $y \geq (h + r)/2$. Now

$$(9.7) \qquad j(y)\xi_{n+1}(y)\phi_0(X,p+m,q) \in L_2^{1,P}(\Omega_{0,n+2})$$

and

$$(9.8) \qquad \xi_n(y)\ \tilde{j}(y)\ R(A_p,z)\ f(X) \in D(A_p^N(\Omega_{0,n+1}))$$

and a second application of the integral identity of (3.19), together with (9.6), give

$$\int_{\Omega} \Delta\, \overline{\{j(y)\phi_0(X,p+m,q)\}}\ \xi_n(y)\ R(A_p,z)\ f(X)\ dX$$

(9.9)

$$= \int_{\Omega} \overline{j(y)\xi_{n+1}(y)\phi_0(X,p+m,q)}\ \Delta\{\xi_n(y)\tilde{j}(y)\ R(A_p,z)\ f(X)\}dX$$

$$= \int_{\Omega} j(y)\ \overline{\phi_0(X,p+m,q)}\ \Delta\{\xi_n(y)\ R(A_p,z)\ f(X)\}dX$$

because $\xi_{n+1}(y) \equiv 1$ on supp ξ_n and $\tilde{j}(y) \equiv 1$ on supp j. Also, Leibniz's rule for distribution derivatives implies

$$(9.10) \qquad \Delta\{\xi_n R(A_p,z)f\} = \xi_n \Delta R(A_p,z)f + 2\xi_n'\ D_y R(A_p,z)f + \xi_n'' R(A_p,z)f.$$

Combining this and the differential equation $\Delta R(A_p,z)f = -A_p R(A_p,z)f = -f - z\ R(A_p,z)f$ gives

$$(\Delta+\omega^2(p+m,q))\ \{\xi_n\ R(A_p,z)f\} = -\xi_n f + (\omega^2-z)\xi_n\ R(A_p,z)f$$

(9.11)

$$+\ 2\xi_n'\ D_y R(A_p,z)f + \xi_n''\ R(A_p,z)f$$

Combining (9.3), (9.9) and (9.11) gives

$$\tilde{f}(p+m,q,z) = (J\,f)_0^{\sim}\ (p+m,q) - \int_{\text{supp } f} \overline{\phi_0(X,p+m,q)}\ \xi_n(y)j(y)f(X)dX$$

(9.12)

$$+\ (\omega^2(p+m,q)-z) \int_\Omega \overline{\phi_0(X,p+m,q)}\ \xi_n(y)j(y)R(A_p,z)f(X)dX$$

$$+\ 2\int_\Omega \overline{\phi_0(X,p+m,q)}\ \xi_n'(y)\ j(y)\ D_y R(A_p,z)f(X)dX$$

$$+\int_\Omega \overline{\phi_0(X,p+m,q)}\ \xi_n''(y)\ j(y)\ R(A_p,z)f(X)dX$$

Now $\xi_n(y) \equiv 1$ on supp f and hence the first two terms of the right-hand side of (9.12) cancel for $n \geq n_0 = n_0(f)$. In view of the definition (3.28), (3.31) of the unitary spectral mapping $\Phi_{0,p}$ associated with $A_{0,p}$, equation (9.12) implies that for all $n \geq n_0$ one has

$$\tilde{f}(p+m,q,z) = (\omega^2(p+m,q) - z)\ \{\Phi_{0,p}(\xi_n J\,R(A_p,z)f)\}_m(q)$$

(9.13)

$$+\ 2\ \{\Phi_{0,p}(\xi_n' J\,D_y\ R(A_p,z)f)\}_m(q)$$

$$+\ \{\Phi_{0,p}(\xi_n'' J\,R(A_p,z)f)\}_m(q).$$

Now $J\ R(A_p,z)f \in L_2(B_0)$ and $J\ D_y\ R(A_p,z)f$ because $R(A_p,z)f \in D(A_p) \subset L_2^1(\Omega)$. Moreover, $0 \leq \xi_n(y) \leq 1$, $\xi_n(y) \to 1$ when $n \to \infty$ for all $y \geq 0$

and supp $\xi_n' \cup$ supp $\xi_n'' \subset \{y \mid n \leq y \leq n+1\}$. It follows by Lebesgue's dominated convergence theorem that $\xi_n\, J\, R(A_p,z)f \to J\, R(A_p,z)f$, $\xi_n'\, J\, D_y\, R(A_p,z)f \to 0$ and $\xi_n''\, J\, R(A_p,z)f \to 0$ in $L_2(B_0)$ when $n \to \infty$. Hence passage to the limit $n \to \infty$ in (9.13) gives

$$\tilde{f}(p+m,q,z) = (\omega^2(p+m,q) - z)\,\{\Phi_{0,p}(J\, R(A_p,z)f)\}_m(q) \tag{9.14}$$

which is equivalent to (5.26).

Proof of Lemma 5.3. This result follows from the continuity of $\tilde{f}(p+m,q,\lambda \pm i\sigma)$ for $q > 0$, $\lambda \in [p^2,\infty) - T_p$ and $\sigma \geq 0$. The details of the proof are precisely the same as in [34, Lemma 6.6] and are therefore not repeated here.

Proof of Lemma 5.4. The starting point for the proof of (5.34) is equation (9.1) with $z = \lambda + i\sigma$, $\lambda \in I \subset [p^2,\infty) - T_p$ and $0 < \sigma \leq \sigma_0$. (9.1) can be *written*

$$\tilde{f}(p+m,q,z) = (J\,f)_0^{\sim}(p+m,q) + g(p+m,q,z) \tag{9.15}$$

where

$$g(p+m,q,z) = \int_{\Omega_{h,r}} \overline{M(X,p+m,q)}\, R(A_p,z)\, f(X)\, dX. \tag{9.16}$$

Note that (see (5.9))

$$M(X,p+m,q) = 2\, D_y\, \{j'(y)\, \phi_0(X,p+m,q)\} - j''(y)\, \phi_0(X,p+m,q) \tag{9.17}$$

and hence

$$g(p+m,q,z) = g_1(p+m,q,z) + g_2(p+m,q,z) \tag{9.18}$$

where

$$(9.19)\qquad g_1(p+m,q,z) = -\int_{\Omega_{h,r}} \overline{\phi_0(X,p+m,q)}\; j''(y)\; R(A_p,z)\; f(X)\; dX$$

and

$$(9.20)\qquad g_2(p+m,q,z) = 2\int_{\Omega_{h,r}} D_y\; \{j'(y)\; \overline{\phi_0(X,p+m,q)}\}\; R(A_p,z)\; f(X)\; dX\ .$$

In the last integral note that $R(A_p,z)f$ is in $L_2^{2,\ell oc}([h,\infty),L_2(-\pi,\pi))$ (cf. Lemma 4.1) while $j'(y)\;\phi_0(X,p+m,q) \in C_0^\infty([h,\infty),L_2(-\pi,\pi))$ and $j(r) = 0$. It follows that

$$(9.21)\qquad g_2(p+m,q,z) = -2\int_{\Omega_{h,r}} \overline{\phi_0(X,p+m,q)}\; j'(y)\; D_y\; R(A_p,z)\; f(X)\; dX\ .$$

Note that (9.19) and (9.21) extend by continuity to $z = \lambda \pm i\sigma$, with $\lambda \in I$ and $0 \le \sigma \le \sigma_0$, by Theorem 4.15.

Equations (9.15) and (9.18) imply that

$$(9.22)\qquad |\tilde{f}(p+m,q,z)|^2 \le 4(|(J\,f)_0^\sim\,(p+m,q)|^2 + |g_1(p+m,q,z)|^2 + |g_2(p+m,q,z)|^2).$$

Moreover, Parseval's relation(3.29) for $A_{0,p}$ implies

$$(9.23)\qquad \sum_{m\in Z}\int_0^\infty |(J\,f)_0^\sim\,(p+m,q)|^2\, dq = \|J\,f\|^2_{L_2(B_0)} \le \|f\|^2_{L_2(\Omega_{0,k})}$$

where supp $f \subset \Omega_{0,k}$. Hence to prove Lemma 5.4 it will suffice to prove (5.34) with $\tilde{f}$ replaced by g_1 and g_2. For g_1, equation (9.19), Parseval's relation (3.29) and Corollary 4.17 imply

$$\sum_{m\in Z} \int_0^\infty |g_1(p+m,q,z)|^2 dq = \| j'' \; R(A_p,z)f\|^2_{L_2(B_0)} \tag{9.24}$$

$$\leq (\text{Max } |j''(y)|)^2 \; \|R(A_p,z)f\|^2_{L_2(\Omega_{0,r})}$$

$$\leq (\text{Max } |j''(y)|)^2 \; C^2 \; \|f\|^2_{L_2(\Omega_{0,k})}$$

for all $z = \lambda \pm i\sigma$ with $\lambda \in I$ and $\sigma \in [0,\sigma_0]$ where $C = C(I,p,\sigma_0,k,r) = C(I,p,\sigma_0,f)$ is the constant of Corollary 4.17. The proof of Lemma 5.4 may be completed by noting that the integral (9.21) for g_2 has the same form as (9.19) but with $j'' \; R(A_p,z)f$ replaced by $2 \; j' \; D_y \; R(A_p,z)f$. An estimate for g_2 of the same form as (9.24) follows because the $L_2(\Omega_{0,r})$ norm of $D_y \; R(A_p,z)f$ is majorized by the $L_2^1(\Delta,\Omega_{0,r})$ norm of $R(A_p,z)f$.

<u>Proofs of Theorems 5.5, 5.6 and 5.7</u>. These results all follow from (5.35) by the spectral theorem and standard Hilbert space methods and therefore will not be given here. A detailed development of these arguments in the case of exterior domains may be found in [34, pp. 109ff].

<u>Proof of Theorem 5.8</u>. Only the orthogonality relation (5.45) need be proved. The proof presented here is based on a method introduced in [34] for the case of exterior domains. The proof for the case of grating domains differs in some important technical details from that of [34] and is therefore presented in full here.

The isometry $\Phi_{\pm,p}$ is known to satisfy (5.45) if and only if [34, p. 116]

$$N(\Phi^*_{\pm,p}) = \{0\} \; ; \tag{9.25}$$

i.e., the null space of $\Phi^*_{\pm,p}$ contains only the zero vector. Equation

(5.45) will be proved by verifying (9.25). The following two lemmas are needed.

Lemma 9.1. For all $h = \{h_m(q)\} \in \Sigma \oplus L_2(R_0)$ one has

$$\Phi^*_{\pm,p}\, h(X) = \underset{M\to\infty}{\text{l.i.m.}} \sum_{|m|\le M} \int_0^M \phi_\pm(X,p+m,q)\, h_m(q)\, dq \tag{9.26}$$

where the convergence is in $L_2(\Omega)$.

Lemma 9.2. Let $h \in N(\Phi^*_{\pm,p})$ and let $\Psi(\lambda)$ be a bounded Lebesgue measurable function on $\lambda \ge p^2$. Then

$$h' = \{\Psi(\omega^2(p+m,q))h_m(q)\} \in N(\Phi^*_{\pm,p}). \tag{9.27}$$

Proofs of Lemmas 9.1 and 9.2. Lemma 9.1 is a direct consequence of (5.38) and (5.42); see [34, Lemma 6.17]. To prove Lemma 9.2 let $f \in L_2(\Omega)$ and note that the definitions of $\Phi_{\pm,p}$ and $\Phi^*_{\pm,p}$ and Theorem 5.7 imply

$$\begin{aligned}
(f,\Phi^*_{\pm,p}\, h') &= (\Phi_{\pm,p}\, f,h') \\
&= \sum_{m\in Z} \int_0^\infty \overline{\tilde{f}_\pm(p+m,q)}\, \Psi(\omega^2(p+m,q))\, h_m(q)\, dq \\
&= \sum_{m\in Z} \int_0^\infty \overline{\overline{\Psi}\, (\omega^2(p+m,q))\, \tilde{f}_\pm(p+m,q)}\, h_m(q)\, dq \\
&= \sum_{m\in Z} \int_0^\infty \overline{(\Phi_{\pm,p}\, \overline{\Psi}(A_p)f)_m(q)}\, h_m(q)\, dq \\
&= (\Phi_{\pm,p}\, \overline{\Psi}(A_p)f,h) = (\overline{\Psi}(A_p)f,\Phi^*_{\pm,p}h) = 0\ .
\end{aligned} \tag{9.28}$$

This proves (9.27) since $f \in L_2(\Omega)$ is arbitrary.

Choice of $\Psi(\lambda)$. Let

$$I = [a,b] \subset [p^2,\infty) - T_p \tag{9.29}$$

and define

$$\Psi(\lambda) = \exp\{-it\,\lambda^{1/2}\}\ \chi_I(\lambda),\ \lambda \geq p^2, \tag{9.30}$$

where $t \in R$ and $\chi_I(\lambda)$ is the characteristic function of I. It will be shown that Lemma 9.2 with this class of functions $\Psi(\lambda)$ implies (9.25). The following notation will be used.

$$N = \{m : \omega^2(p+m,q) \in I \text{ for some } q > 0\}\ . \tag{9.31}$$

Note that N is a finite set. Moreover, $q \to \omega^2(p+m,q)$ is monotone for $q \in R_0$ and hence for each $m \in N$

$$\lambda = \omega^2(p+m,q) \in I \iff q = \sqrt{\lambda - (p+m)^2} \in I_{m,p} \subset R_0 - E_{m,p} \tag{9.32}$$

where $I_{m,p}$ is a compact interval and $E_{m,p}$ is defined by (5.15). With this choice of Ψ, Lemmas 9.1 and 9.2 imply that if $h \in N(\Phi^*_{\pm,p})$ then

$$\begin{aligned} \Phi^*_{\pm,p}\ h'(X) &= \sum_{m\in Z} \int_0^\infty \phi_\pm(X,p+m,q)\ \Psi(\omega^2(p+m,q))\ h_m(q)\,dq \\ &= \sum_{m\in Z} \int_{I_{m,p}} \phi_\pm(X,p+m,q)\ e^{-it\omega(p+m,q)}\ h_m(q)\,dq = 0 \end{aligned} \tag{9.33}$$

in $L_2(\Omega)$. The left hand side of (9.33) defines a solution of the d'Alembert equation in Ω. Its behavior for $t \to \mp\infty$ will be determined and shown to imply (9.25). For this purpose one needs the

Far-Field Form of $\phi_\pm(X,p+m,q)$. This phrase means the form of $\phi_\pm(x,y,p+m,q)$ for large y; i.e., far from the grating. To derive it note that (5.14) and Lemma 4.1 imply that

$$\phi'_{\pm}(X,p+m,q) = \sum_{\ell\in Z} \phi'_{\pm\ell}(y,p+m,q)\ \exp\{i(p+\ell)x\} \tag{9.34}$$

in $L_2^{2,\ell oc}(\Omega_h)$. Moreover, for $y \geq r$

$$\phi'_{\pm\ell}(y,p+m,q) = a_\ell^{\pm}(p+m,q)\ \exp\{i\,y\,w_{p+\ell}(\omega^2(p+m,q) \pm i0)\}\ . \tag{9.35}$$

It follows that for $q \in I_{m,p}$ and $X \in \Omega_r$

$$\begin{aligned}\phi_{\pm}(X,p+m,q) &= \phi_0(X,p+m,q)\\ &+ \sum_{\ell\in L} a_\ell^{\pm}(p+m,q)\ \exp\{i\,x\,p_\ell \pm i\,y\,q_\ell\} + \rho_{\pm}(X,p+m,q)\end{aligned} \tag{9.36}$$

where

$$L = L(p,I) = \{\ell : |p+\ell| < \omega(p+m,q)\}\ , \tag{9.37}$$

and

$$(p_\ell, q_\ell) = (p+\ell, (\omega^2(p+m,q) - (p+\ell)^2)^{1/2}) \tag{9.38}$$

while

$$\rho_{\pm}(X,p+m,q) = \sum_{\ell\in L'} \phi'_{\pm\ell}(y,p+m,q)\ \exp\{i(p+\ell)x\} \tag{9.39}$$

where

$$L' = L'(p,I) = \{\ell : |p+\ell| > \omega(p+m,q)\}\ . \tag{9.40}$$

It is important to note that for $q \in I_{m,p}$ the sets L and L' are independent of q and depend on p and I only. An estimate for the term $\rho_\pm$ in (9.36) is given by

Lemma 9.3. There exists a constant $\mu = \mu(p,I) > 0$ and for each $r' > r$ a constant $C = C(I,p,m,r,r')$ such that

$$(9.41) \qquad |\rho_\pm(X,p+m,q)| \leq C\, e^{-\mu y} \text{ for } X \in \Omega_{r'},\ q \in I_{m,p} .$$

Proof of Lemma 9.3. For brevity write $u(X) = \phi_\pm'(x,p+m,q)$ and note that $u \in F_{p,\zeta,r}$ with $\zeta = \omega^2(p+m,q) \pm i0 \in \overline{M_p^+} - \Sigma_p$. In particular by Lemma 4.1

$$(9.42) \qquad u(X) = \sum_{\ell \in Z} u_\ell(y) \exp\{i(p+\ell)x\} \text{ in } L_2^{2,\ell oc}(\Omega_h)$$

and

$$(9.43) \qquad u_\ell(y) = u_\ell(y') \exp\{-(y-y')((p+\ell)^2 - \omega^2(p+m,q))^{1/2}\}$$

for all $y,y' \geq r$ and all $\ell \in L'$. Now by a Sobolev inequality [1, p. 32] there exists a $C_0 = C_0(h,r)$ such that

$$(9.44) \qquad |u_\ell(r)|^2 \leq C_0^2 \left(\int_h^r |u_\ell'(y)|^2\, dy + \int_h^r |u_\ell(y)|^2\, dy\right)$$

Moreover,

$$(9.45) \qquad \|u(\cdot,y)\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{\ell \in Z} |u_\ell(y)|^2 ,$$

$$(9.46) \qquad \|D_y\, u(\cdot,y)\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{\ell \in Z} |u_\ell'(y)|^2 ,$$

which, with (9.44), imply

$$(9.47)\qquad \begin{aligned} |u_\ell(r)|^2 &\le C_0^2(2\pi)^{-1}(\|D_y u\|^2_{h,r} + \|u\|^2_{h,r}) \\ &\le C_0^2(2\pi)^{-1} \|u\|^2_{1;h,r} \ ; \end{aligned}$$

i.e.,

$$(9.48)\qquad |\phi'_{\pm\ell}(r,p+m,q)|^2 \le C_0^2(2\pi)^{-1} \|\phi'_\pm(\cdot,p+m,q)\|^2_{1;h,r} \ .$$

Now the right hand side of (9.48) is a continuous function of $q \in R_0 - E_{m,p}$ by Theorem 5.1. Thus there exists a constant $C_1 = C_1(I,p,m,r)$ such that

$$(9.49)\qquad |\phi'_{\pm\ell}(r,p+m,q)| \le C_1 \text{ for all } q \in I_{m,p} \ .$$

Next, recalling (9.29), define

$$(9.50)\qquad \mu = \mu(p,I) = \min_{\ell\in L'} \{(p+\ell)^2 - b^2\}^{1/2}$$

so that for all $q \in I_{m,p}$ and $\ell \in L'(p,I)$ one has

$$(9.51)\qquad \{(p+\ell)^2 - \omega^2(p+m,q)\}^{1/2} \ge \{(p+\ell)^2 - b^2\}^{1/2} \ge \mu > 0 \ .$$

Then for $r' > r$ and $X \in \Omega_{r'}$, $q \in I_{m,p}$ one has the estimates

$$(9.52)\qquad \begin{aligned} |\rho_\pm(X,p+m,q)| &\le \sum_{\ell\in L'} |\phi'_{\pm\ell}(y,p+m,q)| \\ &\le \sum_{\ell\in L'} |\phi'_{\pm\ell}(r,p+m,q)| \ \exp\{-(y-r)((p+\ell)^2-(\omega^2(p+m,q))^{1/2}\} \\ &\le C_1 \sum_{\ell\in L'} \exp\{-(y-r)\{(p+\ell)^2 - b^2\}^{1/2}\} \\ &\le (C_1 \sum_{\ell\in L'} \exp\{-(r'-r)\{(p+\ell)^2 - b^2\}^{1/2}\}) \exp\{-\mu(y-r')\} \end{aligned}$$

which implies (9.41).

<u>Proof of Theorem 5.8 (continued)</u>. Substitution of the far-field form (9.36) for $\phi_\pm$ in the identity (9.33) gives the identity

$$u_0(t,X) + u_1(t,X) + u_2(t,X) = 0 \text{ in } L_2(\Omega) \tag{9.53}$$

for all $t \in R$ where

$$u_0(t,X) = \sum_{m\in N} \int_{I_{m,p}} \phi_0(X,p+m,q)\, e^{-it\omega(p+m,q)}\, h_m(q)dq \tag{9.54}$$

$$u_1(t,X) = \sum_{m\in N} \int_{I_{m,p}} \left(\sum_{\ell\in L} a_\ell^\pm(p+m,q)e^{i(xp_\ell \pm yq_\ell)}\right)e^{-it\omega(p+m,q)}\, h_m(q)dq \tag{9.55}$$

$$u_2(t,X) = \sum_{m\in N} \int_{I_{m,p}} \rho_\pm(X,p+m,q)\, e^{-it\omega(p+m,q)}\, h_m(q)dq \tag{9.56}$$

Note that $u_0(t,X)$ has an extension to $X \in B_0$ such that (see (3.32), (3.33))

$$u_0(t,\cdot) = \exp\{-it\, A_{0,p}^{1/2}\}\, h_I^\circ \tag{9.57}$$

where

$$h_I^\circ = \Phi_{0,p}^*\, \{\chi_{m,p}\, h_m : m \in Z\} \in L_2(B_0) \tag{9.58}$$

and $\chi_{m,p}$ is the characteristic function of $I_{m,p}$. In particular, one has

$$\|u_0(t,\cdot)\|^2_{L_2(B_0)} = \|h_I^\circ\|^2_{L_2(B_0)} = \sum_{m\in N} \int_{I_{m,p}} |h_m(q)|^2\, dq \tag{9.59}$$

The proof of Theorem 5.8 will be completed by showing that

$$\lim_{t\to\mp\infty} \|u_0(t,\cdot)\|_{L_2(B_0)} = 0\ . \tag{9.60}$$

It follows from (9.59), (9.60) that $h_m(q) = 0$ for almost all $q \in I_{m,p}$. But $\lambda = \omega^2(p+m,q)$ maps $R_0 - E_{m,p}$ bijectively onto $[p^2,\infty) - T_p$ (see (5.15)). Thus given any $m \in Z$ and any interval $I_{m,p} \subset R_0 - E_{m,p}$ there is an interval $I \subset [p^2,\infty) - T_p$ such that the above relations hold. Thus $h_m(q) \equiv 0$ in $R_0 - E_{m,p}$ for every $m \in Z$, whence $h = 0$ in $\Sigma \oplus L_2(R_0)$ which prove (9.25).

<u>Proof of (9.60)</u>. Consider first the function $u_1(t,X)$ defined by (9.55). It can be written

$$u_1(t,X) = \sum_{m\in N} u_{1,m}(t,X) \tag{9.61}$$

where

$$u_{1,m}(t,X) = \sum_{\ell\in L} u_{1,m,\ell}(t,y) \exp\{i(p+\ell)x\} \tag{9.62}$$

and

$$u_{1,m,\ell}(t,y) = \int_{I_{m,p}} a_\ell^{\pm}(p+m,q)\, e^{\pm iyq_\ell - it\omega(p+m,q)} h_m(q)\,dq \ . \tag{9.63}$$

In the last integral

$$\begin{aligned} q_\ell &= \{\omega^2(p+m,q) - (p+\ell)^2\}^{1/2} \\ &= \{q^2 + (p+m)^2 - (p+\ell)^2\}^{1/2} \equiv Q(q,p+m,p+\ell) \ . \end{aligned} \tag{9.64}$$

Make the change of variable

$$q' = q_\ell = Q(q,p+m,p+\ell) \tag{9.65}$$

in (9.63). Since

$$(9.66) \qquad \omega^2(p+m,q) = \omega^2(p+\ell,q')$$

one has

$$(9.67) \qquad q = Q(q',p+\ell,p+m)$$

and

$$(9.68) \qquad u_{1,m,\ell}(t,y) = \int_{I'_{m,\ell,p}} a_\ell^\pm(p+m,q)\, e^{\pm iyq'-it\omega(p+\ell,q')}\, h_m(q)\, \frac{\partial q}{\partial q'}\, dq'$$

Now each of these integrals has the form of a modal wave in a simple waveguide [35, §5]. Moreover, it was shown in [35] that

$$(9.69) \qquad \lim_{t\to\mp\infty} \| u_{1,m,\ell}(t,\cdot)\|_{L_2(R_0)} = 0.$$

Thus it follows from (9.69)

$$(9.70) \qquad \| u_{1,m}(t,\cdot,y)\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{\ell\in L} |u_{1,m,\ell}(t,y)|^2$$

$$(9.71) \qquad \| u_{1,m}(t,\cdot)\|^2_{L_2(B_0)} = 2\pi \sum_{\ell\in L} \| u_{1,m,\ell}(t,\cdot)\|^2_{L_2(R_0)}$$

and

$$(9.72) \qquad \| u_1(t,\cdot)\|_{L_2(B_0)} \le \sum_{m\in N} \| u_{1,m}(t,\cdot)\|_{L_2(B_0)}$$

that

$$(9.73) \qquad \lim_{t\to\mp\infty} \| u_1(t,\cdot)\|_{L_2(B_0)} = 0 .$$

It will be shown next that the function $u_2(t,X)$ defined by (9.56) satisfies

(9.74) $$\lim_{t\to\mp\infty} \|u_2(t,\cdot)\|_{L_2(\Omega)} = 0 .$$

This is a consequence of the following two lemmas.

Lemma 9.4. The function $u(t,X) = u_2(t,X)$ defined by (9.56) has the properties

(9.75) $$u(t,\cdot) \in L_2(\Omega) \text{ for all } t \in R,$$

(9.76) $$\lim_{t\to\pm\infty} \|u(t,\cdot)\|_{0,k} = 0 \text{ for all } k > r,$$

and there exists a $\mu > 0$ and for each $r' > h$ a constant $C = C(r')$ such that

(9.77) $$|u(t,X)| \le C\, e^{-\mu y} \text{ for all } X \in \Omega_{r'} \text{ and } t \in R.$$

Lemma 9.5. If $u(t,X)$ is any function having properties (9.75), (9.76), (9.77) then

(9.78) $$\lim_{t\to\pm\infty} \|u(t,\cdot)\|_{L_2(\Omega)} = 0 .$$

Proof of Lemma 9.4. To verify (9.75) note that by (9.53), $u(t,\cdot) = u_2(t,\cdot) = -u_0(t,\cdot) - u_1(t,\cdot)$ in $L_2(\Omega)$. But $u_0(t,\cdot) \in L_2(B_0)$ by the spectral theory of $A_{0,p}$ (§3 above) and $u_1(t,\cdot) \in L_2([-\pi,\pi] \times R)$ by the theory of waveguides as developed in [35]. Thus the restrictions of these functions to Ω are in $L_2(\Omega)$.

The decomposition $u = u_2 = -u_0 - u_1$ also implies (9.76) because u_0 and u_1 both represent waves in simple waveguides which have this local decay property; see [35].

Property (9.77) is a consequence of the definition of u_2, equation (9.56), and Lemma 9.3. Indeed, combining (9.41) and (9.56) gives (9.77) with $\mu = \mu(p,I)$ defined by (9.50) and

$$C = C(I,p,m,r,r') \sum_{m\in N} \int_{I_{m,p}} |h_m(q)|\, dq \qquad (9.79)$$

Proof of Lemma 9.5. Conditions (9.75) and (9.77) imply that one has for each $r' > h$ and $k > r'$,

$$\begin{aligned} \|u(t,\cdot)\|^2_{L_2(\Omega)} &= \|u(t,\cdot)\|^2_{0,k} + \|u(t,\cdot)\|^2_{k,\infty} \\ &= \|u(t,\cdot)\|^2_{0,k} + \int_k^\infty \int_{-\pi}^{\pi} |u(t,x,y)|^2\, dxdy \\ &\le \|u(t,\cdot)\|^2_{0,k} + C^2 \int_k^\infty \int_{-\pi}^{\pi} e^{-2\mu y}\, dxdy \\ &= \|u(t,\cdot)\|^2_{0,k} + (\pi\, C^2/\mu)\, e^{-2\mu k} \end{aligned} \qquad (9.80)$$

where $C = C(r')$ is independent of k. Making $t \to \pm\infty$ in (9.80) with k fixed gives, by (9.76),

$$\limsup_{t\to\pm\infty} \|u(t,\cdot)\|^2_{L_2(\Omega)} \le (\pi\, C^2/\mu)\, e^{-2\mu k} \qquad (9.81)$$

for all $k > r'$. This implies (9.78) since the left hand side of (9.81) is independent of k.

Proof of Theorem 5.8 (concluded). The proof may be concluded by verifying (9.60). Now the identity (9.53) implies

$$\|u_0(t,\cdot)\|_{L_2(B_0)} \le \|u_0(t,\cdot)\|_{L_2(B_0-\Omega)} + \|u_0(t,\cdot)\|_{L_2(\Omega)}$$

(9.82)

$$\le \|u_0(t,\cdot)\|_{L_2(B_0-\Omega)} + \|u_1(t,\cdot)\|_{L_2(B_0)} + \|u_2(t,\cdot)\|_{L_2(\Omega)}.$$

Moreover, $B_0 - \Omega$ is bounded and hence $u_0(t,\cdot) \to 0$ in $L_2(B_0 - \Omega)$ by the local decay property for $A_{0,p}$. The remaining terms on the right hand side of (9.82) tend to zero when $t \to \mp\infty$ by (9.73) and (9.74).

§10. Proofs of the Results of §6.

Proof of Theorem 6.1. It will be shown that if $\phi_{\pm}(X,p+m,q)$ are the generalized eigenfunctions for A_p whose existence is guaranteed by Theorem 5.1 then the functions $\psi_{\pm}(X,p,q)$ defined by (6.5) have properties (6.1), (6.2), (6.3). This will prove the existence statement of Theorem 6.1. Note that $q \in \mathcal{E}_{m,p} \Leftrightarrow (p,q) \in \mathcal{E}$ (see (2.30) and (5.15)). Hence the construction (6.5) is valid for $(p,q) \in R_0^2 - \mathcal{E}$.

The sets $D(A^{\ell oc})$ are characterized in the cases of the Neumann and Dirichlet boundary conditions by (see (1.26), (1.28))

$$(10.1) \quad D(A^{N,\ell oc}(G)) = L_2^{1,\ell oc}(\Delta,G) \cap \{u : (1.14) \text{ holds for } v \in L_2^{1,com}(G)\},$$

$$(10.2) \qquad D(A^{D,\ell oc}(G)) = L_2^{1,\ell oc}(\Delta,G) \cap L_2^{D,\ell oc}(G) .$$

As a first step it will be verified that (6.5) defines a function $\psi_{\pm}(\cdot,p,q) \in L_2^{1,\ell oc}(\Delta,G)$ for each $(p,q) \in R_0^2 - \mathcal{E}$. It is clear that $\psi_{\pm}(\cdot,p,q) \in L_2^{\ell oc}(G) \subset \mathcal{D}'(G)$ for $(p,q) \in R_0^2 - \mathcal{E}$ because $\phi_{\pm}(\cdot,p_0+m,q)$ $\in L_2^{\ell oc}(\Omega)$ for $p_0 \in (-1/2,1/2]$, $m \in Z$ and $q \in R_0 - \mathcal{E}_{m,p}$. It remains to show that $\nabla\psi_{\pm}(\cdot,p,q)$ and $\Delta\psi_{\pm}(\cdot,p,q)$, as elements of $\mathcal{D}'(G)$, are also in $L_2^{\ell oc}(G)$. Now by definition $\phi_{\pm}(\cdot,p+m,q) \in L_2^{1,\ell oc}(\Delta,\Omega)$ and hence (6.5) implies

$$(10.3) \qquad \psi_{\pm}(\cdot,p,q) \in L_2^{1,\ell oc}\left(\Delta, \bigcup_{\ell \in Z} \Omega^{(\ell)}\right) .$$

Hence, it is only necessary to verify that $\psi_{\pm}(\cdot,p,q)$, $\nabla\psi_{\pm}$ and $\Delta\psi_{\pm}$ are locally square integrable near the lines $\{(2\ell+1)\pi \times \gamma : \ell \in Z\}$ (see (3.4)). Moreover, $\psi_{\pm}((2\ell+1)\pi \pm 0,y,p,q)$ and $D_1\psi_{\pm}((2\ell+1)\pi \pm 0,y,p,q)$ exist in

$L_2^{\ell oc}(\gamma)$ (see the discussion preceding (3.7)) and the p-periodic boundary condition for $\phi_\pm$ and (6.5) imply

$$(10.4)\quad \begin{cases} \psi_\pm((2\ell+1)\pi+0,\cdot,p,q) = \psi_\pm((2\ell+1)\pi-0,\cdot,p,q) \\ D_1\psi_\pm((2\ell+1)\pi+0,\cdot,p,q) = D_1\psi_\pm((2\ell+1)\pi-0,\cdot,p,q)\ . \end{cases}$$

The proof that $\psi_\pm(\cdot,p,q) \in L_2^{1,\ell oc}(\Delta,G)$ will be completed by proving

Lemma 10.1. The distribution derivatives $D_j\psi_\pm(\cdot,p,q)$ are given by

$$(10.5)\quad D_j\psi_\pm(x,y,p,q) = \exp\{2\pi i\ell p_0\}\, D_j\phi_\pm(x-2\pi\ell,y,p_0+m,q),\ (x,y) \in \Omega^{(\ell)},$$

for $j = 1,2$. Moreover, $\psi_\pm(\cdot,p,q)$ satisfies (6.2) as a distribution on G.

Proof of Lemma 10.1. (10.5) will be proved for $j = 1$. Thus it will be shown that for all $\theta \in C_0^\infty(G)$ one has

$$(10.6)\quad \int_G \psi_\pm\, D_1\theta\, dX = -\int_G D_1\psi_\pm\, \theta\, dX$$

where $D_1\psi_\pm \in L_2^{\ell oc}(G)$ is defined by (10.5). This will be verified for functions θ with supp $\theta \subset \Omega^{(0)} \cup \Omega^{(1)} \cup (\pi \times \gamma)$. In this case (10.6) is a consequence of (10.4) and the equations

$$(10.7)\quad \int_{\Omega^{(0)}} \psi_\pm\, D_1\theta\, dX = -\int_{\Omega^{(0)}} D_1\psi_\pm\, \theta\, dX + \int_\gamma \psi_\pm(\pi-0,y,p,q)\, dy$$

$$(10.8)\quad \int_{\Omega^{(1)}} \psi_\pm\, D_1\theta\, dX = -\int_{\Omega^{(1)}} D_1\psi_\pm\, \theta\, dX - \int_\gamma \psi_\pm(\pi+0,y,p,q)\, dy\ .$$

Equation (10.7) may be verified by calculating

$$(10.9) \qquad \int_{\Omega(0)} \psi_\pm \, D_1(\phi_\delta \theta) \, dX ,$$

where $\phi_\delta(x) = \phi((x-\pi)/\delta)$, $\phi_\delta(x) \equiv 1$ for $x \le \pi-\delta$, $\phi_\delta(x) \equiv 0$ for $x \ge \pi$ and $0 \le \phi_\delta(x) \le 1$, and then making $\delta \to 0$. The technique is explained in [35, p. 57ff]. The case of a general $\theta \in C_0^\infty(G)$ may be proved in the same way. The proof of (10.5) for $j = 2$ is similar. Moreover, an analogous calculation, based on (10.4), gives

$$(10.10) \quad \Delta\psi_\pm(x,y,p,q) = \exp\{2\pi i \ell p_0\} \, \Delta\phi_\pm(x-2\pi\ell, y, p_0+m, q), \quad (x,y) \in \Omega^{(\ell)},$$

and it follows from (5.2) that $\psi_\pm$ satisfies (6.2).

<u>Proof of Theorem 6.1 (continued)</u>. To complete the proof that $\psi_\pm(\cdot,p,q) \in D(A^{\ell oc})$ in the Neumann case, condition (1.14) must be proved for $v \in L^{1,com}(G)$. Now for such a v one has, by Lemma 10.1,

$$(10.11) \qquad \begin{aligned} \int_G \Delta\psi_\pm \, \bar{v} \, dX &= \sum_{\ell \in Z} \int_{\Omega^{(\ell)}} \Delta\psi_\pm \, \bar{v} \, dX \\ &= \sum_{\ell \in Z} \int_{\Omega} \Delta\psi_\pm(x+2\pi\ell, y, p, q) \, \bar{v} \, (x+2\pi\ell, y) \, dX \\ &= \sum_{\ell \in Z} \int_{\Omega} \Delta\phi_\pm(x,y,p,q) \, e^{2\pi i \ell p} \, \bar{v} \, (x+2\pi\ell, y) \, dX \\ &= \int_{\Omega} \Delta\phi_\pm \, \bar{u} \, dX \end{aligned}$$

where

$$(10.12) \qquad u(x,y) = \sum_{\ell \in Z} e^{-2\pi i \ell p} \, v(x+2\pi\ell, y) \in L_2^{1,p,com}(\Omega) .$$

Note that the sums in (10.11), (10.12) are finite because $v \in L_2^{1,com}(G)$. A similar calculation gives

$$\int_G \nabla\psi_\pm \cdot \nabla\bar{v}\, dX = \int_\Omega \Delta\phi_\pm \cdot \nabla\bar{u}\, dX \tag{10.13}$$

and adding (10.11), (10.13) gives

$$\int_G \{\Delta\psi_\pm\, \bar{v} + \nabla\psi_\pm \cdot \nabla\bar{v}\}\, dX = \int_\Omega \{\Delta\phi_\pm\, \bar{u} + \nabla\phi_\pm \cdot \nabla\bar{u}\}\, dX = 0 \tag{10.14}$$

because $\phi_\pm \in D(A^{N,\ell oc}(\Omega))$ and $u \in L_2^{1,p,com}(\Omega)$ (see (3.19)).

To complete the proof that $\psi_\pm(\cdot,p,q) \in D(A^{\ell oc})$ in the Dirichlet case, it must be shown that $\psi_\pm(\cdot,p,q) \in L_2^{D,\ell oc}(G)$ = Closure of $C_0^\infty(G)$ in $L_2^{1,\ell oc}(G)$. This follows immediately from (6.5) because $\psi_\pm(\cdot,p,q)$ is p-periodic and $\phi_\pm(\cdot,p+m,q) \in L_2^{D,p,\ell oc}(\Omega)$ = Closure of $C_p^\infty(\Omega)$ in $L_2^{1,\ell oc}(\Omega)$. To see this note that on any set $K \cap G$ where K is compact in R^2 the functions $\theta \in C_p^\infty(G)$ coincide with functions $\theta' = \phi\theta$ where $\phi \in C_0^\infty(R^2)$ and $\phi(X) \equiv 1$ on K.

It has been shown that $\psi_\pm(\cdot,p,q)$, defined for all $(p,q) \in R_0^2 - E$ by (6.4), satisfies (6.1) and (6.2). Condition (6.3) is also immediate because ψ_0 and ϕ_0 satisfy $\psi_0(X,p,q) = \mathcal{O}^{p_0}\, \phi_0(X,p_0+m,q)$ (see (3.27)) and hence

$$\psi_\pm'(X,p,q) = \mathcal{O}^{p_0}\, \phi_\pm'(X,p_0+m,q)\ . \tag{10.15}$$

It follows that ψ_+' (resp., ψ_-') is an outgoing (resp., incoming) R-B wave for G.

The uniqueness of $\psi_\pm(\cdot,p,q)$ was proved in §6. To complete the proof of Theorem 6.1 the continuity of $(p,q) \to \psi_\pm(\cdot,p,q) \in L_2^{1,\ell oc}(\Delta,G)$

for $(p,q) \in R_0^2 - E$ must be shown. Note that since $\psi_\pm$ satisfies (6.2) it will be enough to prove the continuity of the mapping $(p,q) \to \psi_\pm(\cdot,p,q) \in L_2^{1,\ell oc}(G)$. Thus it must be shown that for each compact $K \subset R^2$ and each $(p_0,q_0) \in R_0^2 - E$ one has

$$(10.16)\qquad \begin{cases} \| \psi_\pm(\cdot,p,q) - \psi_\pm(\cdot,p_0,q_0)\|_{L_2(K\cap G)} \to 0 \\ \\ \| \nabla\psi_\pm(\cdot,p,q) - \nabla\psi_\pm(\cdot,p_0,q_0)\|_{L_2(K\cap G)} \to 0 \end{cases}$$

when $(p,q) \to (p_0,q_0)$. For the functions $\psi_0(\cdot,p,q)$ the continuity conditions (10.16) follow from (1.33), (1.34) by direct calculation. For $\psi_\pm'(\cdot,p,q)$ they follow from (6.5) and the continuity of $(p,q) \to \phi_\pm'(\cdot,p,q) \in L_2^{1,\ell oc}(\Omega)$: i.e.,

$$(10.17)\qquad \begin{cases} \| \phi_\pm'(\cdot,p,q) - \phi_\pm'(\cdot,p_0,q_0)\|_{L_2(K\cap\Omega)} \to 0 \\ \\ \| \nabla\phi_\pm'(\cdot,p,q) - \nabla\phi_\pm'(\cdot,p_0,q_0)\|_{L_2(K\cap\Omega)} \to 0 \end{cases}$$

when $(p,q) \to (p_0,q_0)$. (10.17) is a consequence of Theorem 4.15 and the definitions (5.13) and (5.14). (10.17) and (6.5) imply (10.16) because $K \cap G$ is contained in a finite union of the sets $K \cap \overline{\Omega^{(\ell)}}$.

Proof of Theorem 6.2. This was given in §6.

Proof of Corollary 6.3. As remarked in §6, these results follow from Theorem 6.2 and the fact that $L_2^{com}(G)$ is dense in $L_2(G)$. The details may be found in [34, p. 109] where the corresponding results are proved for exterior domains.

Proof of Theorem 6.4. The proof outlined in §6 will be completed here. The two boundary conditions will be discussed separately.

The Dirichlet Case. Proceeding as in §6, let $f \in L_2^{com}(G)$ and define

$$u = R(A^D(G),z)f \tag{10.18}$$

and

$$v_M(X) = \phi_M(X)\ u(X)\ ,\ x \in G\ , \tag{10.19}$$

where $\phi_M(X) = \psi(|X| - M) \in C_0^2(R^2)$ satisfies $\phi_M(X) \equiv 1$ on G_M and supp $\phi_M \subset G_{M+1}$. Then it is easy to verify that $v_M \in D(A^D(G))$ and

$$\begin{aligned}(A^D(G)-z)\ v_M(X) &= -(\Delta+z)\ \phi_M(X)\ u(X)\\ &= \phi_M(X)\ f(X) - 2\nabla u \cdot \nabla\phi_M - u\ \Delta\phi_M\\ &= f(X) + g_M(X) \text{ for } M \geq M_0(f)\ ,\end{aligned} \tag{10.20}$$

where g_M is defined by (6.21), because $\phi_M(X) \equiv 1$ on supp f for $M \geq M_0(f)$.

Equation (10.20) implies (6.20). To verify (6.22) note that by (6.2) one has

$$\begin{aligned}\hat{v}_{M\pm}(p,q) &= \int_G \overline{\psi_\pm(X,p,q)}\ v_M(X)\ dX\\ &= -\omega^{-2}(p,q) \int_G \Delta\ \overline{\psi_\pm(X,p,q)}\ v_M(X)\ dX\ .\end{aligned} \tag{10.21}$$

Now $\psi_\pm(\cdot,p,q) \in L_2^{1,\ell oc}(\Delta,G)$ and hence

$$\int_G \{(\Delta\overline{\psi}_\pm)\, v_M + \nabla\overline{\psi}_\pm \cdot \nabla v_M\}\, dX = 0 \tag{10.22}$$

because $v_M \in L_2^D(G)$ and supp v_M is compact. Indeed, $v_M = \lim \phi_n$ in $L_2^1(G)$ where $\phi_n \in C_0^\infty(G)$ and (10.22) holds with v_M replaced by ϕ_n by the distribution definitions of $\Delta\psi_\pm$ and $\nabla\psi_\pm$. Similarly, one has

$$\int_G \{\overline{\psi}_\pm(\Delta\, v_M) + \nabla\overline{\psi}_\pm \cdot \nabla v_M\}\, dX = 0 \tag{10.23}$$

because $v_M \in D(A^D(G))$, supp v_M is compact and $\psi_\pm \in L_2^{D,\ell oc}(G)$. Combining (10.21), (10.22) and (10.23) gives

$$\hat{v}_{M\pm}(p,q) = -\omega^{-2}(p,q) \int_G \overline{\psi_\pm(X,p,q)}\, \Delta v_M(X)\, dX\ . \tag{10.24}$$

Finally, combining (10.20) and (10.24) gives

$$\begin{aligned} \hat{v}_{M\pm}(p,q) &= -\omega^{-2}(p,q)(\psi_\pm(\cdot,p,q),\Delta v_M) \\ &= \omega^{-2}(p,q)(\psi_\pm(\cdot,p,q),f+g_M+zv_M) \\ &= \omega^{-2}(p,q)(\hat{f}_\pm(p,q)+\hat{g}_{M\pm}(p,q)+z\hat{v}_{M\pm}(p,q))\ . \end{aligned} \tag{10.25}$$

Solving this equation for $\hat{v}_{M\pm}$ gives (6.22) and hence (6.23).

To find the limiting form of (6.23) for $M \to \infty$ note that $|\phi_M(X)| \le 1$ and $\phi_M(X) \to 1$ for all $X \in G$ when $M \to \infty$. Moreover,

$$\lim_{M\to\infty} g_M = 0 \text{ in } L_2(G) \tag{10.26}$$

because in the definition (6.21) ∇u and u are in $L_2(G)$, $|\nabla\phi_M(I)|$ and $\Delta\phi_M(X)$ are bounded uniformly for all M and supp $g_M \subset G_{M+1} - G_M$. Hence passage to the limit $M \to \infty$ in (6.23) gives (6.18) for $f \in L_2^{com}(G)$. The general case follows by a density argument.

<u>The Neumann Case</u>. The method presented above can be used. However, the definition of the multiplier ϕ_M must be modified to ensure that $v_M \in D(A^N(G))$. If $\phi_M \in C_0^2(\overline{G})$ then it is easy to show that $v_M \in L_2^1(\Delta,G)$. The hypothesis $G \in S$ of §1 will be used to construct a function $\phi_M \in C_0^2(\overline{G})$ such that $v_M = \phi_M u$ also satisfies the Neumann boundary condition. The construction is similar to the one used above to prove Theorem 4.6 in the Neumann case.

To construct ϕ_M let $\sigma(x,y)$, $\tau(x,y)$ be the tangent-normal coordinates defined in the neighborhoods $\mathcal{O} + (2\pi m,0)$ of the points $((2m-1)\pi,y_0)$ as in §8 following (8.177). Define ξ_2 by (8.178) as before and let $\eta_1,\eta_3 \in C^2(R)$ satisfy $0 \le \eta_j(\alpha) \le 1$ and

$$\eta_j(\alpha) = \begin{cases} 1 \text{ for } \alpha \le -\delta_j \\ 0 \text{ for } \alpha \ge \delta_j \end{cases} \tag{10.27}$$

where $\delta_j > 0$. Define

$$\phi_M^1(x,y) = \eta_1(\sigma)\ \xi_2(\tau) + \eta_3(x - (2M+1)\pi)[1 - \xi_2(\tau)] \tag{10.28}$$

for all $(x,y) \in G \cap \{(x,y) : x \ge 0\}$. Note that if $0 < \delta < \pi$ then for δ_1, δ_2, δ_3 small enough one has

$$\phi_M^1(x,y) = \begin{cases} 1 \text{ for } x \le (2M+1)\pi - \delta\ , \\ 0 \text{ for } x \ge (2M+1)\pi + \delta\ . \end{cases} \tag{10.29}$$

Extend ϕ_M^1 to the rest of G by

(10.30) $$\phi_M^1(x,y) = 1 - \phi_M^1(-x,y) \text{ for } (x,y) \in G \cap \{(x,y) : x \le 0\} .$$

Finally let $\phi_M^2(y) \in C^2(R)$ satisfy $0 \le \phi_M^2(y) \le 1$, $\phi_M^2(y) \equiv 1$ for $y \le M$, $\phi_M^2(y) \equiv 0$ for $y \ge M + 1$ and define

(10.31) $$\phi_M(x,y) = \phi_M^1(x,y)\ \phi_M^2(y) .$$

Then ϕ_M has the desired properties. It is clear that $\phi_M \in C_0^2(\overline{G})$ and

(10.32) $$\operatorname{supp} \phi_M \subset \{(x,y) : -(2M+1)\pi - \delta \le x \le (2M+1)\pi + \delta, 0 \le y \le M + 1\} .$$

Moreover, in the strip $|x - (2M + 1)\pi| \le \delta$, $0 \le y \le h$, one has $\xi_2(\tau) \equiv 1$ and hence $\phi_M(x,y) = \eta_1(\sigma(x,y))$. Similarly, in $|x + (2M + 1)\pi| \le \delta$, $0 \le y \le h$ one has $\phi_M(x,y) = 1 - \eta_1(\sigma(x,y))$. This property implies that $v_M = \phi_M u$ satisfies the Neumann boundary condition on Γ; see (8.184). The remainder of the proof of Theorem 6.4 is the same as in the Dirichlet case.

Proof of Theorem 6.5. It was remarked in §6 that (6.24) and (6.26) are direct consequences of (6.27) (see [34, p. 110]). Relation (6.27) will be derived from Theorem 6.4 and Stone's formula. The latter states that if $I = [a,b] \subset R$ then for all $f \in L_2(G)$ one has

(10.33) $$\frac{1}{2}(f, [\Pi(b) + \Pi(b-) - \Pi(a) - \Pi(a-)]f) = \lim_{\sigma \to 0+} \frac{\sigma}{\pi} \int_I \| R(A, \lambda + i\sigma) f \|^2 \, d\lambda .$$

Now Theorem 6.4 and Fubini's theorem imply that

$$\frac{\sigma}{\pi}\int_I \|R(A,\lambda+i\sigma)f\|^2\, d\lambda = \frac{\sigma}{\pi}\int_I \int_{R_0^2} \frac{|\hat{f}_\pm(p,q)|^2}{|\omega^2(p,q)-\lambda-i\sigma|^2}\, dp\, dq\, d\lambda$$

(10.34)

$$= \int_{R_0^2} \left(\frac{\sigma}{\pi}\int_I \frac{d\lambda}{(\lambda-\omega^2(p,q))^2+\sigma^2}\right) |\hat{f}_\pm(p,q)|^2\, dp\, dq\ .$$

Moreover, if

$$K(\sigma,p,q) = \frac{\sigma}{\pi}\int_I \frac{d\lambda}{(\lambda-\omega^2(p,q))^2+\sigma^2} \tag{10.35}$$

then $0 \le K(\sigma,p,q) \le 1$ for all $(p,q) \in R_0^2$ and $\sigma > 0$ and $\lim K(\sigma,p,q) = \chi_I(\omega^2(p,q))$ for $\sigma \to 0$; [34, p. 98]. Hence (10.33) and (10.34) imply

$$\frac{1}{2}\,(f,[\Pi(b)+\Pi(b-)-\Pi(a)-\Pi(a-)]f) = \int_{R_0^2} \chi_I(\omega^2(p,q))\,|\hat{f}_\pm(p,q)|^2\, dp\, dq \tag{10.36}$$

by Lebesgue's dominated convergence theorem. On making $a \to b$ in (10.36) and using the relation $\Pi((b-)-) = \Pi(b-)$ one finds that $\Pi(b) = \Pi(b-)$ for all $b \in R$. Then putting $\Pi(b-) = \Pi(b)$, $\Pi(a-) = \Pi(a)$ in (10.36) gives (6.27).

Proof of Theorem 6.6. This result can be proved by the method used for the case of exterior domains in [34, p. 113]. The multiplier ϕ_m of [34, p. 114] may be replaced by the function ϕ_M used to prove Theorem 6.4. The remaining details are the same as in [34] and will not be repeated here.

Proof of Theorem 6.7. It will suffice to prove the relation (6.32), or equivalently

$$\|\Phi_{\pm}^{*}f\| = \|f\| \ , \tag{10.37}$$

for all $f \in C_0^{\infty}(R_0^2 - E)$.

As a first step, note that for all $f(p,q) \in L_2(R_0^2)$ one has

$$(\Phi_{\pm}^{*}f)(X) = L_2(G)\text{-}\lim_{M\to\infty} \int_{D_M} \psi_{\pm}(X,p,q)\, f(p,q)\, dp\, dq \ . \tag{10.38}$$

The simple proof is the same as for the case of exterior domains [34, p. 117]. If $f \in C_0^{\infty}(R_0^2 - E)$ then (10.38) can be written

$$\begin{aligned}(\Phi_{\pm}^{*}f)(X) &= \int_{R_0^2} \psi_{\pm}(X,p,q)\, f(p,q)\, dp\, dq \\ &= \sum_{m\in Z} \int_{B_0} \psi_{\pm}(X,p+m,q)\, f(p+m,q)\, dp\, dq\end{aligned} \tag{10.39}$$

and only a finite number of terms in the sum are non-zero. In particular, the definition (6.5) of $\psi_{\pm}$ implies that

$$(\Phi_{\pm}^{*}f)(X) = \sum_{m\in Z} \int_{B_0} e^{2\pi i \ell p}\, \phi_{\pm}(x-2\pi\ell,y,p+m,q)\, f(p+m)\, dp\, dq \tag{10.40}$$

for $X \in \Omega^{(\ell)}$.

Next note that

$$\begin{aligned}\|\Phi_{\pm}^{*}f\|^2 &= \int_G |\Phi_{\pm}^{*}f(X)|^2\, dX = \sum_{\ell\in Z} \int_{\Omega^{(\ell)}} |\Phi_{\pm}^{*}\, f(X)|^2\, dX \\ &= \sum_{\ell\in Z} \int_{\Omega} |\Phi_{\pm}^{*}f(x+2\pi\ell,y)|^2\, dx\, dy \ .\end{aligned} \tag{10.41}$$

Now (10.40) implies that for $(x,y) \in \Omega$

$$
\begin{aligned}
(\Phi_\pm^* f)(x+2\pi\ell,y) &= \sum_{m\in Z} \int_{-1/2}^{1/2} \left(\int_0^\infty e^{2\pi i\ell p} \phi_\pm(x,y,p+m,q) f(p+m,q)\,dq\right) dp \\
&= \int_{-1/2}^{1/2} e^{2\pi i\ell p} (\Phi_{\pm,p}^* \{f(p+\cdot,\cdot)\})(X)\,dp
\end{aligned} \tag{10.42}
$$

by Lemma 9.1. The interchange of summation and integration is elementary because the sum is finite for $f \in C_0^\infty(R_0^2 - E)$. Equation (10.42) states that the left hand side of the equation, as a function of $\ell \in Z$, is the set of Fourier coefficients of the function of p defined by $\Phi_{\pm,p}^* \{f(p+\cdot,\cdot)\}$. Thus Parseval's relation for Fourier series implies

$$
\sum_{\ell\in Z} |\Phi_\pm^* f(x+2\pi\ell,y)|^2 = \int_{-1/2}^{1/2} |(\Phi_{\pm,p}^* \{f(p+\cdot,\cdot)\})(X)|^2\,dp \tag{10.43}
$$

Integrating (10.43) over $X \in \Omega$ and using (10.41) gives, again by Fubini's theorem,

$$
\begin{aligned}
\|\Phi_\pm^* f\|^2 &= \int_{-1/2}^{1/2} \int_\Omega |(\Phi_{\pm,p}^* \{f(p+\cdot,\cdot)\})(X)|^2\,dX\,dp \\
&= \int_{-1/2}^{1/2} \|\Phi_{\pm p}^* \{f(p+\cdot,\cdot)\}\|^2\,dp\ .
\end{aligned} \tag{10.44}
$$

Now the orthogonality property for $\Phi_{\pm,p}^*$, Theorem 5.8, implies that

$$
\begin{aligned}
\|\Phi_{\pm,p}^* \{f(p+\cdot,\cdot)\}\|^2 &= \|\{f(p+\cdot,\cdot)\}\|^2 \\
&= \sum_{m\in Z} \int_0^\infty |f(p+m,q)|^2\,dq\ .
\end{aligned} \tag{10.45}
$$

Combining (10.44) and (10.45) gives

(10.46) $$\|\Phi_{\pm}^{*} f\|^2 = \sum_{m \in Z} \|f\|^2_{L_2(B_0+(m,o))} = \|f\|^2_{L_2(R_0^2)}$$

which is equivalent to (10.37).

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13. ABSTRACT

Abstract

This report is the second part of a study of Rayleigh-Bloch (R-B) wave expansions for plane diffraction gratings. The principal concepts and results of the study, including the definition and construction of the R-B waves, the formulation of the corresponding R-B wave expansion theorem and the main ideas for the proofs were given in part I. The present part II contains complete proofs of the results whose proofs were omitted or only sketched in part I.

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